

ASTR 2500 - Ch.3

Kepler's laws (+ Galileo's relative frame of reference velocity transformations) were postulates, not part of deeper law

Newton discovered calculus, postulated the inverse square law of gravity + 3 laws of motion, inventing classical mechan.

- Laws:
- ① \vec{v} const. until force acts
 - ② \vec{F} induces an \vec{a} of m^{-1} ; $\vec{F} = m\vec{a}$
 - ③ forces equal & opposite

Gravity: $F = -\frac{GMm}{r^2}$ direction const
always attractive

Kepler's 2nd law: sweeping out equal areas in equal times

Ang. mom. is $\vec{L} = \vec{r} \times \vec{p}$, $\vec{p} = m\vec{v}$

As a planet moves, its \vec{L} changes

$$\frac{d\vec{L}}{dt} = \frac{d}{dt} (\vec{r} \times \vec{p}) = \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{\vec{v} \times (m\vec{v})} + \vec{r} \times \underbrace{\frac{d\vec{p}}{dt}}_{m\vec{a}}$$

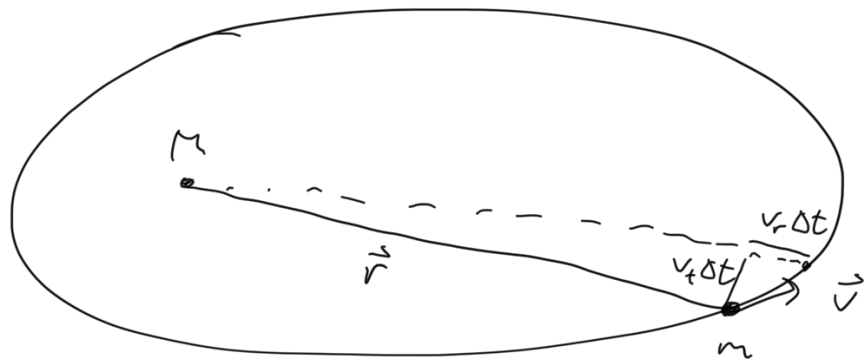
$$= m(\underbrace{\vec{v} \times \vec{v}}_{=0}) + \vec{r} \times \vec{F} \quad m \frac{d\vec{v}}{dt} = \vec{F}$$

Why?

If \vec{F} is a central force $\vec{r} \parallel \vec{F}$
 so that $= 0$ too

Thus $\frac{d\vec{L}}{dt} = 0$: Ang. Mom. conserved
 (both dir. + mag.)

Kepler's 2nd law proved!



Approximate as 2 triangles

$$\Delta A = \frac{1}{2} r v_e \Delta t + \frac{1}{2} v_e v_r \Delta t^2$$

Choose small Δt , 2nd term $\ll 1$ st term

$$\Delta A \approx \frac{1}{2} r v_e \Delta t$$

or $\frac{\Delta A}{\Delta t} \approx \frac{1}{2} r v_e$; taking the limit

$$\text{as } \Delta t \rightarrow 0, \quad \frac{\Delta A}{\Delta t} = \frac{dA}{dt} = \frac{1}{2} r v_e$$

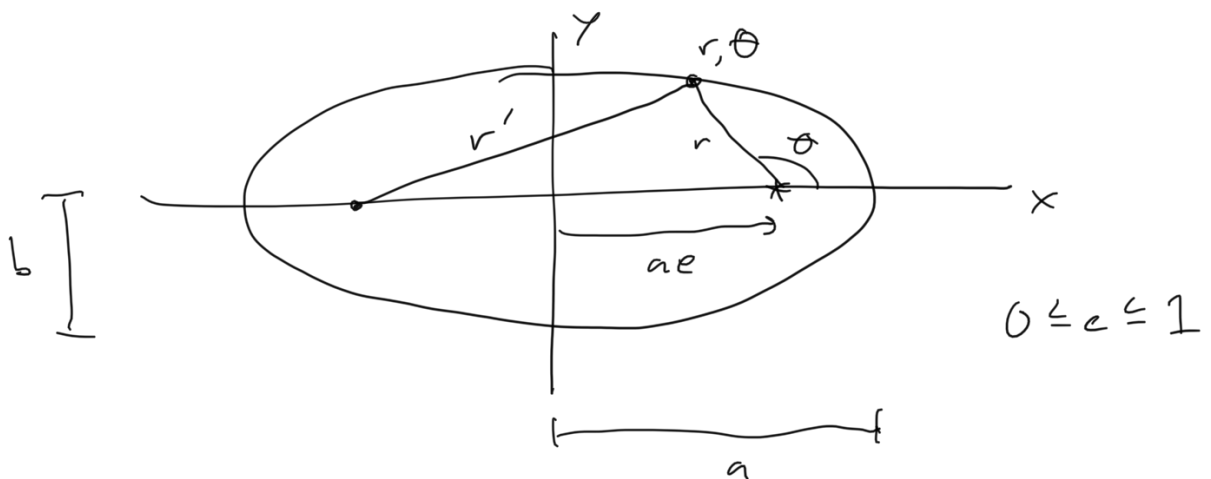
Bring L into this, $\frac{dL}{dt} = r m v \approx r m v_e$

so $\frac{dA}{dt} = \frac{1}{2} \frac{L}{m} \rightarrow \text{const.}$, so $\frac{dA}{dt} = \text{const.}$

Origin of Kepler's 1st law

No time to derive (see 3.1.2),
 but find that the path of a planet
 under an inverse square law, in
 polar coordinates, is

$$r = \frac{L^2}{GMm^2(1 + e \cos \theta)}$$



$$r + r' = 2a \quad (y=0, \quad r = a - ae + r' = a + ae)$$

$$b^2 = a^2(1 - e^2) \quad (x=0, \quad r = r' = a \quad \text{so})$$

$$a^2 e^2 + b^2 = a^2$$

Looking @ these definitions & doing
some math, find

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{L^2/GMm^2}{1+e\cos\theta}$$

$$\text{so } a(1-e^2) = L^2/GMm^2$$

Orbits can have wildly diff. e , but
share the same L

3rd law, $P^2 = k a^3$

from earlier, saw that $\frac{dA}{dt} = \frac{L}{2m}$

$$\text{Over 1 orbit, } \frac{A(\text{ellipse})}{P} = \frac{L}{2m} = \frac{\pi ab}{P}$$

$$b^2 = a^2(1-e^2), \text{ so}$$

$$\frac{L}{2m} = \frac{\pi a^2 (1-e^2)^{1/2}}{P}$$

$$(1-e^2)^{1/2} = \frac{L}{\pi a^2}$$

but also $a(1-e^2) = L^2/GMm^2$ so

$$\frac{L}{2m} = \frac{\pi a^2 \frac{L}{a(1-e^2)}}{P} = \frac{\pi a^2 \frac{L}{L^2/GMm^2}}{P}$$

$$P = a^{3/2} \frac{2\pi}{(GM)^{1/2}} \quad \text{or}$$

$$p^2 = \frac{4\pi}{GM} a^3 = k a^3 \quad \checkmark$$

In reality, the larger mass also moves in an ellipse about the center of mass (focus), so really

$$p^2 = \frac{4\pi}{G(M+m)} a^3$$

Energetics : $E = K + U = \text{const.}$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

- more work, find $e = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$

$e=0$ \rightarrow circular orbit, E negative

$E < 0$ if $e < 1 \rightarrow$ bound orb.its

b/c too little E to escape $|U| > K$

$e=1$ $\rightarrow E=0$, exactly enough E
to escape

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = 0$$

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

↳ path is parabolic ($y \propto x^2$)

$e > 1 \rightarrow E > 0$, unbound orbits

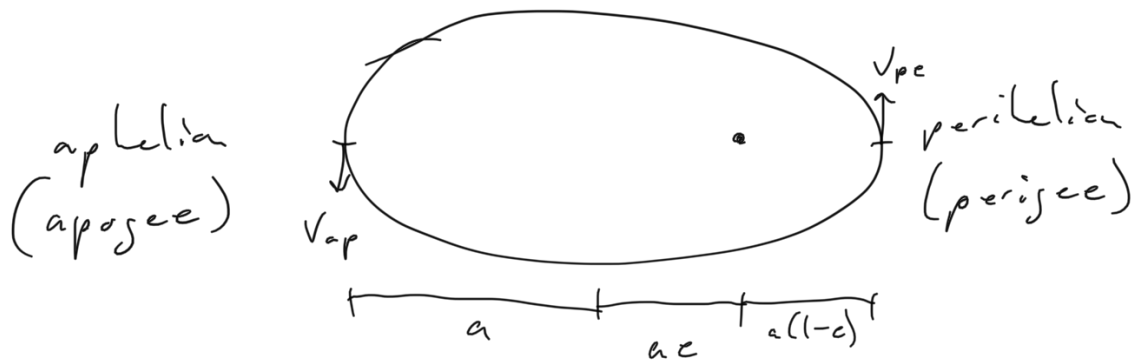
$K > |U|$, hyperbolic path

What if you want to know the speed as a function of time?

- no simple equation, but is for vL

$$v^2 = GM \left(\frac{2}{r} - \frac{1}{a} \right) \quad \text{vis viva equation}$$

Special places in the orbit, closest and farthest from star



$$v_{pe}^2 = GM \left(\frac{2}{a(1-e)} - \frac{1}{a} \right)$$

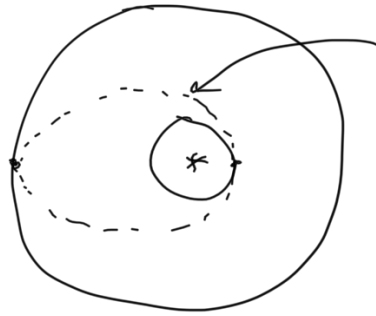
$$= GM \left(\frac{2 - (1-e)}{a(1-e)} \right) = \underline{GM \frac{1+e}{1-e}}$$

$$V_{pe} = \sqrt{\frac{GM}{a} \left(\frac{1+e}{1-e} \right)}$$

$$V_{ap} = \sqrt{\frac{GM}{a} \frac{1-e}{1+e}}$$

For spacecraft sent to other planets, can use a "transfer orbit" so that fuel can be conserved during travel

Hohmann transfer orbit



Be careful of timing, imposes launch window

Virial Theorem

→ one of the most important facts about gravitating systems!

"virial" is a made-up word not in spellcheckers, defined as $\sum \vec{F}_i \cdot \vec{r}_i$ (\vec{r}_i relative to center of mass)

For bound systems, particles can exchange energy, but on average, the total

kinetic + potential \bar{E} 's follow

$$2\langle K \rangle + \langle U \rangle = 0$$

Can be hard to measure the masses of large objects (star clusters, galaxies, clusters of galaxies), but this relation allows mass (potential \bar{E}) to be estimated from motions (kinetic \bar{E})