ASTR 3070 - Week 02b (Orbits)

Kepler's laws (+ Galileo's relative frame of reference velocity transforma were postulates, not part of deeper law

Newton discovered calculus, postulated the inverse square law of gravity + 3 laws of motion, inventing classical mechan

Laws: (1) $\vec{v}$ const. until force acts

(2) $\vec{F}$ induces an $\vec{a} \propto m^{-1}$: $\vec{F} = m$

(3) forces equal + opposite

Gravity: $F = -\dfrac{GMm}{r^2}$ direction cent always attractive

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Kepler's 2^{nd} law: sweeping out equal areas in equal times

Ang. mom. is $\vec{L} = \vec{r} \times \vec{p}$, $\vec{p} = m\vec{v}$

As a planet moves, its $\vec{L}$ changes

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$= 0$ $\quad v \times m(v) \quad m\frac{dv}{dt} = \ldots$

Skip

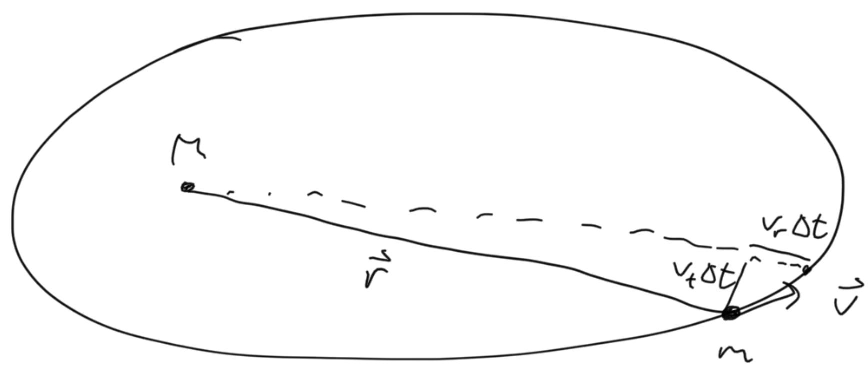
$$= m\underbrace{(\vec{v}\times\vec{v})}_{=0} + \vec{r}\times\vec{F}$$

Why?

If $\vec{F}$ is a central force $\vec{r} \parallel \vec{F}$
so that $= 0$ too

Thus $\frac{d\vec{L}}{dt} = 0$: Ang. Mom. conser.
(both dir. + mag.)

Kepler's 2^{nd} law proved!



Approximate as 2 triangles

$$\Delta A = \frac{1}{2} r v_t \Delta t + \frac{1}{2} v_t v_r \Delta t^2$$

Choose small Δt, 2^{nd} term << 1^{st} term

$$\Delta A \approx \frac{1}{2} r v_t \Delta t$$

or $\frac{\Delta A}{\Delta t} \approx \frac{1}{2} r v_t$; taking the limit

as $\Delta t \to 0$, $\frac{\Delta A}{\Delta t} = \frac{dA}{dt} = \frac{1}{2} r v_t$

$\Delta t \quad dt$

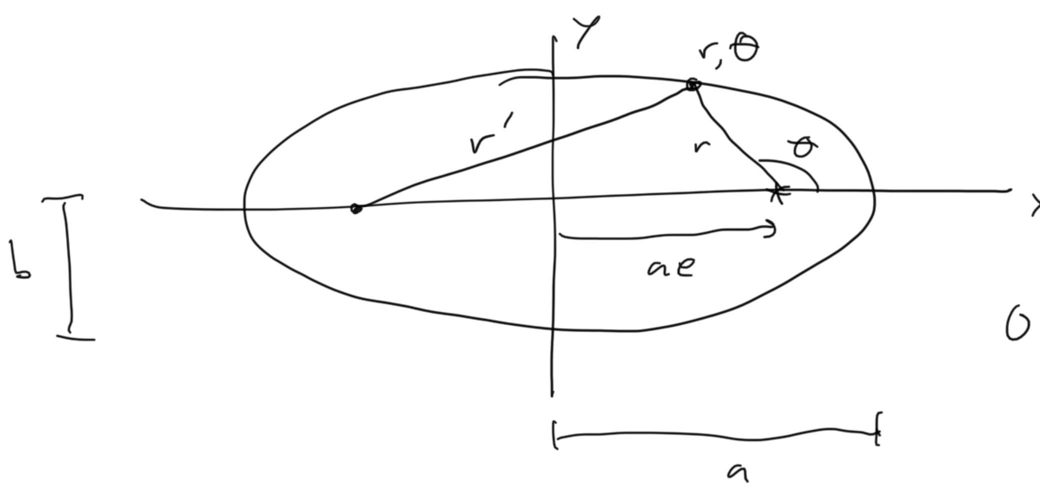
Bring L into this, $|\vec{L}| = rmv \approx rmv_t$

so $\frac{dA}{dt} = \frac{1}{2}\frac{L}{m}$ → consts, so $\frac{dA}{dt} =$ co

Origin of Kepler's 1ST law

No time to derive (see 3.1.2), but find that the path of a planet under an inverse square law, in polar coordinates, is

$$r = \frac{L^2}{GMm^2(1 + e\cos\theta)}$$



$0 \le e \le 1$

$r + r' = 2a$ ($y = 0$, $r = a - ae$ & $r' = a + ae$

$b^2 = a^2(1 - e^2)$ ($x = 0$, $r = r' = a$ so

$$a^2e^2 + b^2 = a^2$$

Looking @ these definitions & doing some math, find

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{L^2/GMm^2}{1+e\cos\theta}$$

so $\quad a(1-e^2) = L^2/GMm^2$

Orbits can have wildly diff. e, but share the same L

3rd law, $\quad P^2 = k a^3$

from earlier, saw that $\frac{dA}{dt} = \frac{L}{2m}$

Over 1 orbit, $\frac{A(\text{ellipse})}{P} = \frac{L}{2m} = \frac{\pi a b}{P}$

$b^2 = a^2(1-e^2)$, so

$$\frac{L}{2m} = \frac{\pi a^2 (1-e^2)^{1/2}}{P} \qquad (1-e^2)^{1/2} = \frac{L}{(GM}$$

but also $a(1-e^2) = L^2/GMm^2$ so

$$\frac{\cancel{L}}{2\cancel{m}} = \frac{\pi a^2 \frac{\cancel{L}}{(GMa)^{1/2}\cancel{m}}}{P}$$

$$P = a^{3/2}\frac{2\pi}{(GM)^{1/2}} \qquad \text{or}$$

$$p^2 = \frac{4\pi^2}{GM} a^3 = k a^3$$

In reality, the larger mass also moves in an ellipse about the center of mass (focus), so really

$$\boxed{p^2 = \frac{4\pi^2}{G(M+m)} a^3}$$

Energetics: $E = K + U = \text{const.}$

$$E = \frac{1}{2} m v^2 - \frac{GMm}{r}$$

- more math, find $\quad e = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$

$\underline{e = 0}$ → circular orbit, E negative

$E < 0$ if $e < 1$ → bound orbits

b/c too little E to escape $|U| > K$

$\underline{e = 1}$ → $E = 0$, exactly enough E to escape

$$E = \frac{1}{2} m v^2 - \frac{GMm}{r} = 0$$

$$\boxed{v_{esc} = \sqrt{\frac{2GM}{r}}}$$

↳ path is parabolic ($y \propto x^2$ 

$\underline{e > 1}$ → $E > 0$, <u>unbound orbits</u>

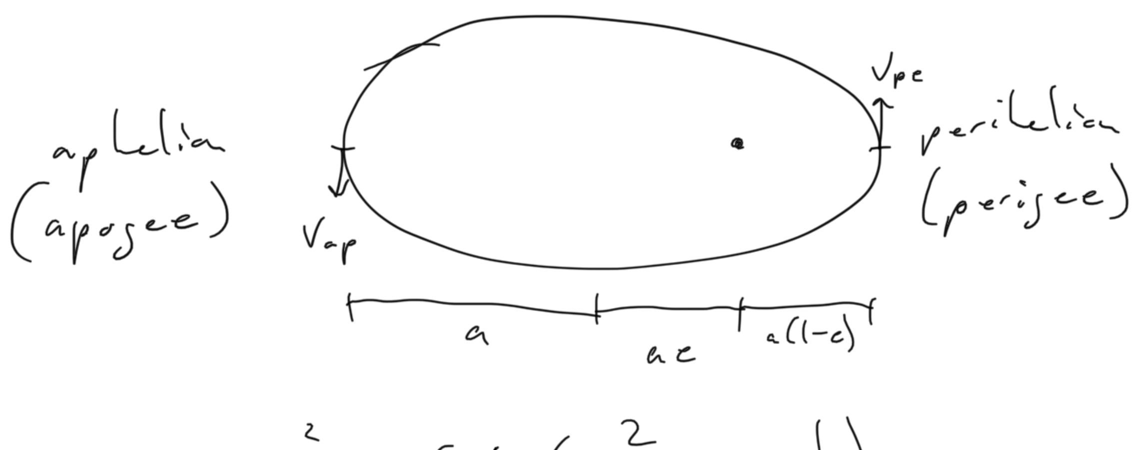
$K > |U|$, hyperbolic path

What if you want to know the speed as a function of time?

- no simple equation, but is for $v(r)$

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$ vis viva equation

Special places in the orbit, closest and farthest from star



$$v_{pe}^2 = GM\left(\frac{2}{a(1-e)} - \frac{1}{a}\right)$$

$$= GM\left(\frac{2-(1-e)}{a(1-e)}\right) = \frac{GM}{} \frac{1+e}{}$$

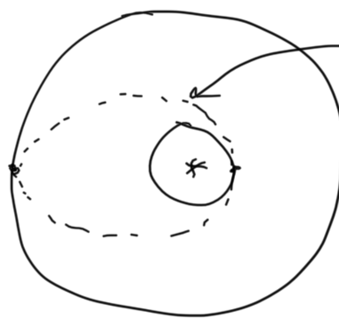
$a \quad 1-e$

$$\boxed{v_{pe} = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}}$$

$$\boxed{v_{ap} = \sqrt{\frac{GM}{a}\,\frac{1-e}{1+e}}}$$

For spacecraft sent to other planets, can use a "transfer orbit" so that fuel can be conserved during travel

Hohmann transfer orbit



ellipse

Be careful of timing, imposes <u>launch window</u>

<u>Virial Theorem</u>

→ one of the most important facts about gravitating systems!

"virial" is a made-up word not in spellcheckers, defined as $\sum \vec{F}_i \cdot \vec{r}_i$

($\vec{r}_i$ relative to center of mass)

For bound systems, particles can exchange energy, but on <u>average</u>, the total kinetic + potential E's follow

$$\boxed{2\langle K \rangle + \langle U \rangle = 0}$$

Can be hard to measure the masses of large objects (star clusters, galaxies, clusters of galaxies), but this relation allows mass (potential E) to be estim from motions (kinetic E)