ASTR 4080 - Week 13

"Overdensities" lead to collapse

static + homogeneous

$$\ddot{R} = -\frac{G(\Delta M)}{R^2}$$

→ extra mass over avg. field

$$\bar{\rho} \equiv \frac{M}{\frac{4}{3}\pi R^3}$$

$$\bar{\rho}(1+\delta) = \frac{M+\Delta M}{\frac{4}{3}\pi R^3} \rightarrow \bar{\rho}\,\delta = \frac{\Delta M}{\frac{4}{3}\pi R^3}$$

$$\boxed{\frac{\ddot{R}}{R}(t) = -\frac{4\pi G\bar{\rho}}{3}\,\delta(t)}$$

extra mass causes collapse (intuitive)

$R(t)$, $\delta(t)$ → need another eqn.

☆ What relates mass + radius?

$$M = \frac{4\pi}{3}\left[\bar{\rho}(1+\delta)\right] R^3 \rightarrow \text{conserved}$$

$$M(t_1) = M(t_2)$$

$$[1+\delta(t_1)]R(t_1)^3 = [1+\delta(t_2)]R(t_2)^3$$

define $R_0 = R(t_1)$ when $\delta(t_1) \rightarrow 0$

$$R(t) = R_0\left[1+\delta(t)\right]^{-1/3}$$

Taylor expand: $R(t) \approx R_0\left[1 - \frac{1}{3}\delta(t)\right]$

for $\delta(t) << 1$

Two derivs yield $\dot{R} = -\frac{R_0}{3}\dot{\delta}$

$$\ddot{R} = -\frac{R_0}{3}\ddot{\delta} \approx -\frac{R}{3}\ddot{\delta}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G\bar{\rho}}{3}\delta = -\frac{1}{3}\ddot{\delta} \rightarrow \ddot{\delta} = 4\pi G\bar{\rho}\,\delta$$

$\rightarrow$ integrating: $\delta(t) = A_1 e^{t/t_{dyn}} + A_2 e^{-t/t_{dyn}}$

$$t_{dyn} = \frac{1}{(4\pi G\bar{\rho})^{1/2}} \approx 10^{hr}\left(\frac{\bar{\rho}}{1\,\text{kg m}^{-3}}\right)^{-1/2}$$

★ how long it takes overdensity to collapse

Jeans Length

→ any perturbation will grow exponentially, faster in a denser environment

BUT: will be offset by pressure

$$P = w \mathcal{E}$$

non-rel.: $w \approx \frac{kT}{\mu c^2}$ ($\sim 10^{-12}$ in this room)

Pressure ↑ as sphere collapses, able to support ↑ gravity <u>IF</u> pressure gradient has <u>enough time</u> to establish itself

↳ able to communicate collapse is happening (& equilibrate)

– can't happen faster than sound speed

$$t_p \sim \frac{R}{c_s}, \quad c_s = c\left(\frac{dP}{d\mathcal{E}}\right)^{1/2} = c\sqrt{w}$$

$$t_{dyn} > t_p$$

can define the size where this occurs: $t_p \sim t_{dyn}$, $t_p = \frac{\lambda_J}{c_s}$

so $\lambda_J \sim c_s t_{dyn} \sim c_s \left(\frac{1}{G\bar{\rho}}\right)^{1/2} \sim c_s \left(\frac{c^2}{G\bar{\varepsilon}}\right)^{1/2}$

fluctuations larger than this collapse

$$\boxed{\lambda_J = 2\pi c_s t_{dyn}}$$

(air: $\lambda_J \sim 10^5$ m)

But this is static case → our universe is expanding!

$$t_{expand} \sim H^{-1} = \left(\frac{3c^2}{8\pi G \bar{\varepsilon}}\right)^{1/2}$$

$$t_{dyn} = \left(\frac{c^2}{4\pi G \bar{\varepsilon}}\right)^{1/2} = \left(\frac{2}{3}\right)^{1/2} H^{-1}$$

$$\boxed{\lambda_J = 2\pi \left(\frac{2}{3}\right)^{1/2} \sqrt{w}\, \frac{c}{H}}$$

For baryons, can convert to the Jeans mass

$$M_J = \rho_{bary} \left(\frac{4}{3}\pi \lambda_J^{\,3}\right)$$

Before decoupling, baryons were dragged along by radiation, so λ_J & M_J determined by $w = \frac{1}{3} \rightarrow M_J \sim 10^{19} M_\odot$

After decoupling, $w = \sqrt{\frac{kT}{mc^2}} << \frac{1}{3}$

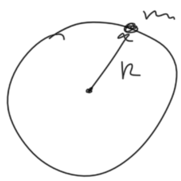
$\sim 10^{-5}$

so $M_J \sim 10^5 M_\odot \rightarrow$ fluctuations larger than present-day star clusters can collapse gravitationally

To follow the growth of perturbations δ, need to include expansion since timescale $1/H$ is comparable to t_{dyn}

<u>Newtonian analogy</u> (sphere again)

$$\ddot{R} = -\frac{GM'}{R^2} = -\frac{G(M+\Delta M)}{R^2} = -\frac{4\pi}{3}G\bar{\rho}R(1+\delta)$$



$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}G\bar{\rho} - \frac{4\pi}{3}G\bar{\rho}\,\delta$$

Again, have mass conservation so

$$R(t) = R_0\left[1+\delta\right]^{-1/3} \propto \bar{\rho}(t)^{-1/3}\left[1+\delta(t)\right]^{-1/3}$$

$$\bar{\rho} \propto a^{-3} \longrightarrow \boxed{R(t) \propto a(t)\left[1+\delta(t)\right]^{-1/3}}$$

Two derivs again: $\dot{R} = \dot{a}(1+\delta)^{-1/3} - \frac{1}{3}a\dot{\delta}(1+\delta)^{-4/3}$

$$\ddot{R} = \ddot{a}(1+\delta)^{-1/3} - \frac{1}{3}\dot{a}\dot{\delta}(1+\delta)^{-4/3} - \frac{1}{3}\dot{a}\dot{\delta}(1+\delta)^{-4/3} - \frac{1}{3}a(1+\delta)^{-4/3}\left[\ddot{\delta} - \frac{4}{3}\dot{\delta}^2(1+\delta)^{-1}\right]$$

$$\ddot{R} = \ddot{a}\frac{R}{a} - \frac{2}{3}\dot{a}\dot{\delta}\frac{R}{a}(1+\delta)^{-1} - \frac{1}{3}\ddot{\delta}R(1+\delta)^{-1} + \frac{4}{9}\dot{\delta}^2(1+\delta)^{-2}R$$

$1+\delta \approx 1$, $\dot{\delta}^2 < \dot{\delta}$ so set

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4\pi}{3}G\bar{\rho} - \frac{4\pi}{3}G\bar{\rho}\delta$$

($\delta = 0$, get accel. from Ch. 4)

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\bar{\rho} \rightarrow \text{subtract that}$$

$$-\frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4\pi}{3}G\bar{\rho}\delta$$

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}\delta$$

rel. version: $\boxed{\ddot{\delta} + 2H\dot{\delta} = \frac{4\pi G}{c^2}\bar{\varepsilon}_m\delta}$

$\delta \rightarrow$ matter component, but in a universe evolving w/ multiple components

using critical density $\rightarrow$ $\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2\delta = 0$

Radiation epoch : $\Omega_m \ll 1$ & $a \propto t^{1/2}$

so $H = \frac{\dot{a}}{a} = \frac{1}{2} t^{-1}$ & find

$$\ddot{\delta} + \frac{1}{t}\dot{\delta} = 0$$

$$\delta(t) = B_1 + B_2 \ln t$$

☆ slow growth of fluctuations by DM (which isn't coupled to rad)

Matter epoch : $\Omega_m \approx 1$, $H = \frac{2}{3t}$

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0$$

$$\delta(t) \simeq \underbrace{D_1 t^{2/3}}_{\text{important after some growth}} + D_2 t^{-1}$$

$$\underline{\delta \propto t^{2/3} \propto a(t) \propto \frac{1}{1+z}}$$

$(\delta \ll 1)$

DM starts @ $z_{rm} = 3440$, while baryons have to wait until $z_{dec} \sim 1090$

☆ Slides

ASTR 4080 - Week 12