

## ASTR 4080 - Week 4

Unknowns :  $\epsilon(t)$ ,  $a(t)$ ,  $P(\epsilon)$

↳ also  $K/R_0$  : fundamental constants,  
possibly determined by constants

Friedmann Eq.:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \epsilon - \frac{Kc^2}{R_0^2 a^2}$$

Fluid Eq.:

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0$$

Eq. of State:  $P = w \epsilon$

(put on left side board)

Slide 2 Whatever is in universe,  
the E. densities simply sum

\* Fluid Eq. holds for each component  
separately then

On right board

$$\dot{\epsilon}_i + 3 \frac{\dot{a}}{a} (\epsilon_i + p) = 0$$

$$\dot{\epsilon}_i + 3 \frac{\dot{a}}{a} \epsilon_i (1 + w_i) = 0$$

$$\frac{1}{\epsilon_i} d\epsilon_i = - \frac{3}{a} (1 + w_i) da$$

Assuming  $w_i \rightarrow \text{const.}$ , integrate

$$\ln \epsilon_i = -3(1 + w_i) \ln a + C$$

$$\boxed{\epsilon_i(a) = \epsilon_{i,0} a^{-3(1+w_i)}}$$

Already know  $w$  for non-rel.,  
rel., &  $\Lambda$  components

$$\underline{w=0} \quad \underline{w=\frac{1}{3}} \quad \underline{w=-1}$$

$$\Sigma_m = \Sigma_{m,0} / a^3 \quad \Sigma_r = \Sigma_{r,0} / a^4 \quad \Sigma_\Lambda = \Sigma_{\Lambda,0}$$

★ refer to expansion on slide

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On right board

$$\frac{\Sigma_\Lambda(a)}{\Sigma_m(a)} = \frac{\Sigma_{\Lambda,0}}{\Sigma_{m,0} a^{-3}} = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} a^3$$

$$a = 0.77 \text{ @ } \Sigma_\Lambda = \Sigma_m$$

$$\frac{\Sigma_m(a)}{\Sigma_r(a)} = \frac{\Sigma_{m,0} a^{-3}}{\Sigma_{r,0} a^{-4}} = \frac{\Omega_{m,0}}{\Omega_{r,0}} a = \frac{1}{3400}$$

$$1+z = \frac{1}{a}, \text{ so } z_{m\Lambda} = 0.31 \text{ \& } z_{rm} \approx 3400$$
$$z_{gal} \sim 10, \quad z_{cmb} \sim 1000$$

Convenient to write Friedmann Eq.

as 
$$\dot{a}^2 = \frac{8\pi G}{3c^2} \sum_i \epsilon_{i,0} a^{-1-3w_i} - \frac{Kc^2}{R_0^2}$$

★ write on left board

w/this formulation, can solve for  $a(t)$   
for different kinds of universes

★ Briefly to Summary slide then Eyring slide  
then back to board

$$\dot{a}^2 = -\frac{Kc^2}{R_0^2} \rightarrow K = -1 \quad (+1 \text{ forbidden})$$

- if spacetime flat, no expansion just SK

otherwise,  $a = \pm \frac{c}{R_0} t + \boxed{a(t) = t/t_0}$

[HW problem w/  $\Gamma = 1$ ]

- Observe light from distant galaxy, here

$$1+z = \frac{1}{a(t_e)} = \frac{t_0}{t_e}$$

$$t_e = \frac{H_0^{-1}}{1+z}$$

What distance is the galaxy at?

$$d_p(t_0) = a(t_0) \int_0^r dr = r$$

↳ @ current time

Light follows null geodesic:  $ds^2 = -c^2 dt^2 + \overset{a(t)^2}{dr^2} = 0$

$$c \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^r dr = r$$

Thus  $d_p(t_0) = c \int_{t_e}^{t_0} \frac{dt}{a(t)}$  ★ Generally true

Empty: simple  $\Rightarrow d_p(t_0) = \frac{c}{H_0} \ln(1+z)$

- Can see farther than Hubble sphere!  
but not distance when photon emitted

$$d_p(t_e) = \frac{1}{1+z} d_p(t_0)$$

Horizon distance  $\rightarrow$  most distant obj. can see

equiv. to  $d_p(t_0)$  as  $z \rightarrow \infty$   $\rightarrow d_{hor}(t_0) = c \int_0^{t_0} \frac{dt}{a(t)} = ct_0 \int_0^{t_0} \frac{dt}{t} = ct_0 [\ln t_0 - \ln 0] = \infty$

★ of course nothing to see if empty...

★ stay on Empty slide, save solutions on board boxed in yellow

$$K=0 \rightarrow \boxed{\ddot{a}^2 = \frac{8\pi G \epsilon_0}{3c^2} a^{-(1+3w)}}$$

< math happens >

$$\boxed{a(t) = \left(\frac{t}{t_0}\right)^{\frac{2/3}{1+w}}}$$

$$\boxed{t_0 = \frac{1}{1+w} \sqrt{\frac{c}{6\pi G \epsilon_0}}}$$

if  $w \neq -1$

$$\rightarrow \ddot{a}^2 = \frac{8\pi G \epsilon_0}{3c^2} a^2 \rightarrow \dot{a} = H_0 a$$

$$\boxed{a(t) = e^{H_0(t-t_0)}}$$

flat  
Λ-only

For matter,  $w \approx 0$  and

$$\boxed{a(t) = \left(\frac{t}{t_0}\right)^{2/3}}$$

Radiation,  $w = \frac{1}{3}$  +

$$\boxed{a(t) = \left(\frac{t}{t_0}\right)^{1/2}}$$

★ Next slide

$$\epsilon_{crit,0} = \frac{3c^2}{8\pi G} H_0^2$$

$$H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \frac{2}{3} \frac{1}{(1+w)} t_0^{-1}$$

$$\text{so } \epsilon_{crit,0} = \frac{c^2 t_0^{-2}}{6\pi (1+w)^2}$$

Matter : Einstein - de Sitter Universe

$$\hookrightarrow t_0 = \frac{2}{3} \frac{1}{H_0}$$

$$\begin{aligned} d_p(t_0) &= c \int_{t_0}^{t_0} \frac{dt}{(t/t_0)^{2/3}} = 3ct_0 \left[ 1 - \left( \frac{t_0}{t_0} \right)^{1/3} \right] \\ &= \frac{2c}{H_0} \left[ 1 - \frac{1}{\sqrt{1+z}} \right] \end{aligned}$$

Radiation :  $t_0 = \frac{1}{2} \frac{1}{H_0}$

$$\begin{aligned} d_p(t_0) &= c \int_{t_0}^{t_0} \frac{dt}{(t/t_0)^{1/2}} = 2ct_0 \left[ 1 - \left( \frac{t_0}{t_0} \right)^{1/2} \right] \\ &= \frac{c}{H_0} \frac{2}{(1+z)} \end{aligned}$$

Lambda : de Sitter Universe

$$t_0 \rightarrow \infty ! \quad a(t=0) = e^{-H_0 t_0} \equiv 0$$

$$\begin{aligned} d_p(t_0) &= c \int_{t_0}^{t_0} e^{H_0(t_0-t)} dt = \frac{c}{H_0} \left[ e^{H_0(t_0-1)} - 1 \right] \\ &= \frac{c}{H_0} 2 \end{aligned}$$

★ Next Slide  $\rightarrow$  discuss then come back here

## Multi-component Universes

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3c^2} \Sigma - \frac{Kc^2}{R_0^2 a^2}$$

$$\Sigma(t) = \sum_i \Sigma_i(t)$$

In terms of critical density, becomes

$$\frac{K}{R_0^2} = \frac{H_0^2}{c^2} (\Omega_0 - 1) \quad @ \quad t = t_0$$

$$\Sigma_{crit,0} \equiv \frac{3c^2 H_0^2}{8\pi G}$$

Sub these 2  
into

so Friedmann Eq. becomes:

$$\frac{H(t)^2}{H_0^2} = \frac{\Sigma(t)}{\Sigma_{crit,0}} + \frac{1 - \Omega_0}{a(t)^2}$$

$$\Omega_i \equiv \frac{\Sigma_i(t)}{\Sigma_{crit,0}}, \quad \Sigma_i(a) = \Sigma_{i,0} a^{-3(1+w_i)}$$

$$\text{so } \frac{H^2}{H_0^2} = \frac{\dot{a}^2}{a^2} \frac{1}{H_0^2} = \sum \Omega_i + \frac{1 - \Omega_0}{a^2}$$

$$\Omega_0 = \sum \Omega_{i,0} + \Omega_i = \Omega_{i,0} a^{-3(1+w_i)}$$



$$\text{so } H_0^{-1} \dot{a} = \left[ \Omega_{r,0} a^{-2} + \Omega_{m,0} a^{-1} + \Omega_{\Lambda,0} a^2 + (1 - \Omega_0) \right]^{1/2}$$

$$\text{And } \int_0^a \frac{da}{Ea} = \int_0^t H_0 dt$$

$$t = \frac{1}{H_0} \int_0^a \frac{da}{E(a)}$$

No analytic solution in the general case, but certain limits are solvable ( $\Omega_{i,0} = 0$ )

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Matter + Curvature  $K=0$  already solved

$$K=+1, \quad a_{\max} = \frac{\Omega_0}{\Omega_0 - 1} \rightarrow \underline{\Omega_0 > 1}$$

Friedmann Eq. is  $\frac{\dot{a}^2}{H_0^2} = \frac{\Omega_0}{a} + (1 - \Omega_0)$

curvature given via  $1 - \Omega_0 = -\frac{Kc^2}{H_0^2 a^2}$

↳ determined by  $\Omega_0$

$$t = \frac{1}{H_0} \int_0^a \frac{da}{E(a)} = \frac{1}{H_0} \int_0^a \frac{da}{\sqrt{\Omega_0 a^{-1} + (1 - \Omega_0)}}$$

Can solve for  $\Omega_0 \neq 1$  case (already solved)

★ Next Slide



In the  $\Omega_{\Lambda,0} > 0$  case, can integrate for an analytic solution

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln \left[ \left( \frac{a}{a_{m\Lambda}} \right)^{3/2} + \sqrt{1 + \left( \frac{a}{a_{m\Lambda}} \right)^3} \right]$$

where  $a_{m\Lambda} = \left( \frac{\Omega_{m,0}}{\Omega_{\Lambda,0}} \right)^{1/3} = \left( \frac{\Omega_{m,0}}{1-\Omega_{m,0}} \right)^{1/3}$

At the time when  $\Omega_m = \Omega_\Lambda$

At early times, b/c  $\epsilon_m \propto a^{-3}$  &  $\epsilon_\Lambda = \text{const}$ , matter dominates when  $a \ll a_{m\Lambda}$

$$\ln[\ ] \rightarrow \ln \left[ 1 + \left( \frac{a}{a_{m\Lambda}} \right)^{3/2} \right] \approx \left( \frac{a}{a_{m\Lambda}} \right)^{3/2}$$

so  $H_0 t \approx \frac{2}{3\sqrt{1-\Omega_{m,0}}} \left( \frac{a^{3/2}}{\left( \frac{\Omega_{m,0}}{1-\Omega_{m,0}} \right)^{1/3}} \right)^{3/2}$

$$\Rightarrow a(t) \approx \left( \frac{3}{2} \Omega_{m,0}^{1/2} H_0 t \right)^{2/3} \propto t^{2/3}$$

as expected for a flat, matter-dominated universe

When  $a \gg a_{\text{ML}}$  ( $\Lambda$  dominated)

$$\ln[\ ] \approx \ln\left[2\left(\frac{a}{a_{\text{ML}}}\right)^{3/2}\right] = \ln 2 + \frac{3}{2} \ln \frac{a}{a_{\text{ML}}}$$
$$\approx \frac{3}{2} \ln \frac{a}{a_{\text{ML}}}$$

so  $e^{H_0 t \sqrt{1-\Omega_{m,0}}} \approx \frac{a}{a_{\text{ML}}}$

or  $a(t) = a_{\text{ML}} e^{H_0 \sqrt{1-\Omega_{m,0}} t}$

$a(t) \propto e^{kt}$   $\rightarrow$  flat,  $\Lambda$ -dominated

Equation also gives  $t_0$  (for  $a_0 = 1$ )

$$t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln \left[ \frac{\sqrt{1-\Omega_{m,0}} + 1}{\Omega_{m,0}^{1/2}} \right]$$

$\rightarrow \Omega_{m,0} \sim 0.3, \Omega_{\Lambda,0} \sim 0.7$  so

$$t_0 \sim 0.96 H_0^{-1} = 0.96 t_H$$

$H_0 = 70$  :  $t_H = 14.6 \text{ yr} + t_0 = 13.5 \text{ Gyr}$

$H_0 = 68$  :  $t_H = 14.56 \text{ yr} + t_0 = 13.9 \text{ Gyr}$

$\Omega_{m,0} = 0.31 / \Omega_{\Lambda,0} = 0.69 \rightarrow \boxed{t_0 = 13.7 \text{ Gyr}}$

Add Curvature  $\rightarrow$  no analytic solution

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## Radiation + Matter

Early times,  $\epsilon_r$  insignificant, but  $\epsilon_r \propto a^{-4}$   
faster than  $\epsilon_m$  so do same exercise  
for rad + matter

$$\text{Flat: } \frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}$$

$H = \frac{\dot{a}}{a}$ , rearrange to integrate

$$H_0 dt = \frac{a da}{\Omega_{r,0}^{1/2} \sqrt{1 + a/a_{\text{arr}}}}$$

$$a_{\text{arr}} = \frac{\Omega_{r,0}}{\Omega_{m,0}} \rightarrow \text{integrate}$$

$$H_0 t = \frac{2 a_{\text{arr}}^2}{3 \Omega_{r,0}^{1/2}} \left[ 1 - \left(1 - \frac{a}{2 a_{\text{arr}}}\right) \left(1 + \frac{a}{a_{\text{arr}}}\right)^{1/2} \right]$$

So does the equation behave correctly when we dominate?

Matter:  $a \gg a_{\text{rm}}$

$$H_0 t \propto \left[ 1 - \left( -\frac{1}{2a_{\text{rm}}} \right) \left( \frac{a}{a_{\text{rm}}} \right)^{1/2} \right] a^{3/2}$$

thus  $a \propto t^{2/3} \rightarrow$  same as before

Radiation:  $a \ll a_{\text{rm}}$

$$\left( 1 + \frac{a}{a_{\text{rm}}} \right)^{1/2} \approx 1 + \frac{1}{2} \frac{a}{a_{\text{rm}}}$$

$$\left( 1 - \frac{a}{2a_{\text{rm}}} \right) \left( 1 + \frac{a}{2a_{\text{rm}}} \right) = 1 + \left( \frac{a}{2a_{\text{rm}}} \right)^2$$

$$H_0 t \propto a^2 \rightarrow a \propto t^{1/2}$$

Can solve for time radiation = matter

$$t_{\text{rm}} = 50 \text{ kyr}$$

So, radiation epoch short  $\rightarrow$  for long term evolution of  $a$ , can be ignored

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