Homework 7

Due **March 19 at 6pm via Canvas**

Please show all work, writing solutions/explanations clearly, or no credit will be given. You are encouraged to work together, but everyone must turn in independent solutions: do not copy from others or from any other sources.

1. Suppose it were suggested that black holes of mass $M_{\rm BH}$ made up all the dark matter in the halo of our galaxy. Derive a rough expression for how far away you would expect the nearest such black hole to be and for how frequently you would expect a black hole to pass within 1 AU of the Sun. Don't try to derive an exact expression; instead, do an order-of-magnitude estimate. What are the estimates for this MACHO-type dark matter if $M_{\rm BH}$ is equal to the mass of the Earth? For what $M_{\rm BH}$ would there always be about one dark matter black hole within Earth's orbit?

2. Imagine that the universe were made entirely of dark matter, helium, and relativistic matter. In other words, imagine that the baryonic content of the universe consisted of only helium rather than hydrogen as was previously assumed in the derivation of CMB formation. The ionization energy of helium is $Q = 24.6$ eV, the energy required to remove one electron from a neutral helium atom. At what temperature would the fractional ionization of helium be $X = 0.5$? Assume that $\eta = 6 \times 10^{-10}$ and that each helium atom is only ionized with charge +1. The relevant statistical weight factor for the ionization of helium is $g_{\rm He}/(g_e g_{\rm He^+}) = 1/4$.

3. The hot gas in a galaxy cluster is observed to have a power law density profile $\rho(r) \propto r^{-\Gamma}$, where Γ is a constant, and an exponentially declining temperature profile $T \propto e^{-r/r_t}$, where r_t is a scale radius (also a constant) that is smaller than the total radius of the cluster. Write down the expression $M(< r)$ in terms of r and constants. Are these profiles for density and temperature physically plausible (i.e., will the function $M(< r)$ always return physically sensible numbers)?

4. We know from observations that the intergalactic medium is currently ionized. Thus, at some time between recombination epoch $t_{\rm rec}$ and present day t_0, the intergalactic medium must have been reionized. The fact that we can see small fluctuations in the CMB places limits on how early the reionization took place. Assume that the baryonic component of the universe instantaneously became completely reionized at some time t_*. For simplicity, assume that the universe is spatially flat with matter density parameter $\Omega_{\rm m,0}$ and cosmological constant density parameter $\Omega_{\Lambda,0}$, and that the baryonic component of the universe is pure hydrogen with baryon density parameter $\Omega_{\rm b,0} = 0.048$. The Thomson scattering optical depth τ of reionized material is

$$\tau = \sigma_e c \int_{t_*}^{t_0} n_e(t) dt \,, \tag{1}$$

where $\sigma_e = 6.65 \times 10^{-29}$ m^2 is the Thomson scattering cross-section.

(a) Make the variable change from t to z (with t_* corresponding to z_*) and express $n_e(z)$ in terms of $\Omega_{\rm b,0}$ and other cosmological parameters and physical constants. In the end, write down the full integral expression of the optical depth $\tau(z_*)$, which should include $\Omega_{\rm b,0}$, $\Omega_{\rm m,0}$, H_0, c, and $m_{\rm H}$ (hydrogen atom mass. For our purpose, $m_{\rm H} \approx m_p$, the proton mass).

(b) The contribution to τ will be dominated by electrons at high redshifts, so for practical purposes, you can simply drop $\Omega_{\Lambda,0}$ in the expression in (a). With this simplification, do the integral and find the expression of $\tau(z_*)$.

(c) The inferred $\tau(z_*)$ from observation of the CMB anisotropy by the Planck satellite is $\tau(z_*) = 0.07$. For $H_0 = 70$ km s^{-1} Mpc^{-1}, $\Omega_{\rm m,0} = 0.3$, and $\Omega_{\rm b,0} = 0.048$, figure out z_*.

(d) Given the cosmological parameters and z_* in (c), what is the age of the universe when reionization occurred? [Hint: As in (b), you can again drop $\Omega_{\Lambda,0}$ and neglect the radiation component in the age calculation.]