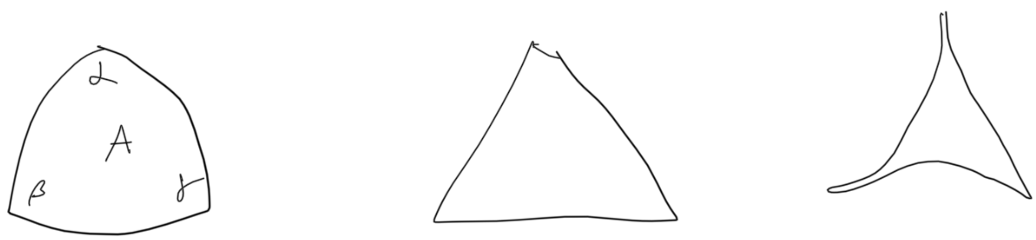
ASTR 4080 -Week 3

★ Ask how $\underline{\text{curvature}}$ can be measured

$\boxed{\text{ANS}}$ angle sum PLUS area (for R_0)

$$\alpha + \beta + \gamma = \pi + \frac{\kappa A}{R_0^2}$$

$\kappa = +1$ $\qquad$ $\kappa = 0$ $\qquad$ $\kappa = -1$



Objects appear different sizes depending on geometry:

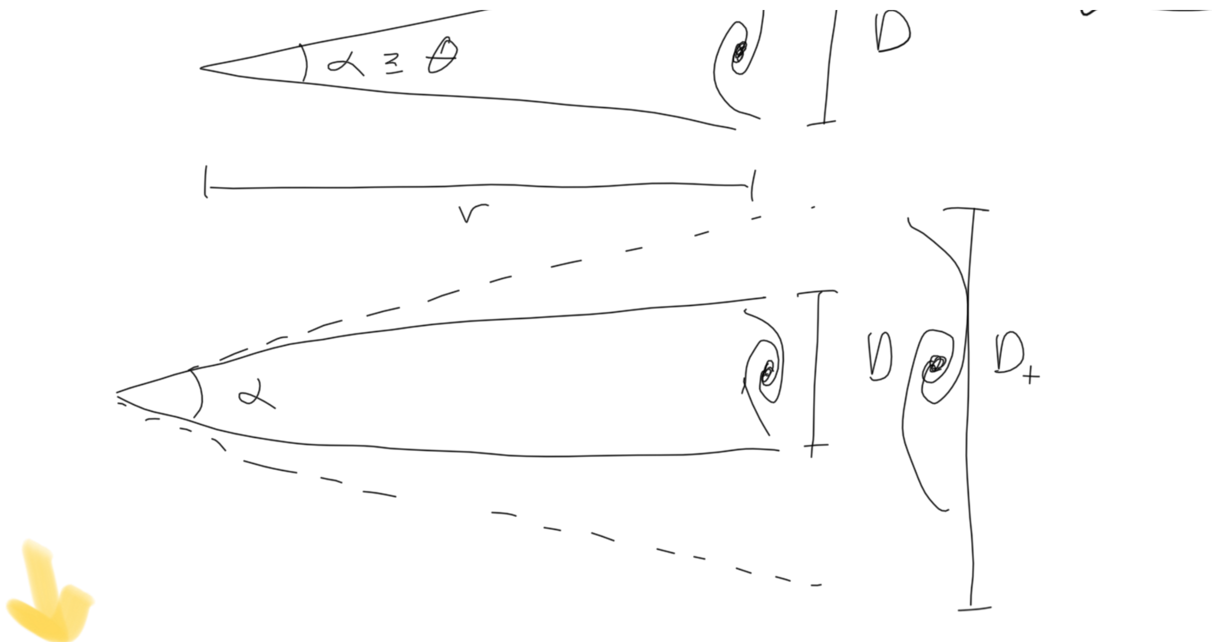
$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_\kappa(r)^2 d\Omega^2 \right]$$

→ path light takes, $ds = 0$,
consider instant of time, $dt = 0$

Skip?

$$\Delta r = \pm S_\kappa(r) \Delta\theta \quad (\text{b/c } d\phi = 0 \text{ also})$$

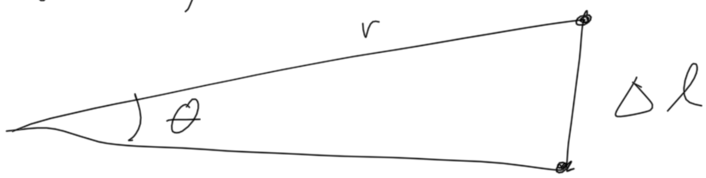
$$D = \pm S_\kappa(r)\, \alpha \quad , \quad \boxed{\alpha_{\text{flat}} = \frac{D}{r}} \quad (\text{small angle approx}$$


α
D
r
α
D
D+

$$d\ell^2 = dr^2 + S_K(r)^2 \, d\Omega^2$$

$d\Omega^2 \to d$

$\phi = 0$, $\theta = 0 \to$ small #



$$\Delta\ell^2 = \Delta r^2 + S_K(r)^2 \Delta\theta^2$$

$\Delta r = 0$, $\Delta\ell = S_K(r)\,\Delta\theta$

$$\boxed{\Delta\theta = \frac{\Delta\ell}{S_K(r)}}$$

ONLY for SMALL ANGLES since other have to

INTEGRAT

$K=0$: $\alpha = \frac{D}{r}$

$K=+1$: $\alpha = \frac{D}{R_0 \sin r/R_0} > \frac{D}{r}$

$K=-1$: $\alpha = \frac{D}{R_0 \sinh r/R}$ $< D/r$

$\downarrow$

$\sinh x = e^x/2$ so

$\alpha \approx \frac{2D}{R_0} e^{-r/R}$

★ Galaxies don't appear anomalously ↑ or ↓, so

$R_0 > r$, v. high z (Hubble sphere $\to c/H_0 \sim 4$

SKIP to Friedmann Eq.

Einstein's Field Equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein tensor   stress-energy tensor

What is a <u>tensor</u>?

nxn array (like a matrix) but

can contain operators, etc.

$G_{\mu\nu} \rightarrow$ curvature of spacetime (4 dimensions

$T_{\mu\nu} \rightarrow$ energy dist. of in spacetime, causing curvature

also

$$G_{00} = \frac{8\pi G}{c^4} T_{00}$$

$$G_{10} = \frac{8\pi G}{c^4} T_{10} \quad \text{etc.} \rightarrow G_{44} = \frac{8\pi G}{c^4} T_{44}$$

symmetric, so $G_{10} = G_{01}$, etc. 10 indep. equations

nonlinear, 2^{nd} order diff. eq.s

What is $G_{\mu\nu}$? More tensors!

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$$R_{\mu\nu} = \partial_\eta\left(\Gamma_\mu{}^\eta{}_\nu\right) - \partial_\nu\left(\Gamma_\eta{}^\eta{}_\mu\right) + \Gamma_\eta{}^\eta{}_\lambda \Gamma_\mu{}^\lambda{}_\nu - \Gamma_\mu{}^\lambda{}_\eta \Gamma_\nu{}^\eta{}_\lambda$$

$$\Gamma_\alpha{}^\gamma{}_\beta = \frac{1}{2} g^{\delta\gamma}\left[\partial_\beta(g_{\alpha\beta}) + \partial_\alpha(g_{\beta\delta}) - \partial_\delta(g_{\alpha,\right.$$

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} \quad (\text{variable name is } x^\alpha)$$

$g_{\alpha\beta}$ (metric tensor) $\rightarrow$ distance b/t points

$$ds^2 = \sum g_{\alpha\beta}\, dx_\alpha\, dx_\beta$$

☆ Clearly, becomes v. complicated v. quickly

For us, Robertson-Walker metric is what matters

$$\boxed{ds^2 = -c^2 dt^2 + a(t)^2\left[dr^2 + S_K(r)^2 d\Omega^2\right.}$$

Friedmann Equation

→ links $a(t)$, k, R_0, $\varepsilon(t)$

Newtonian derivation lets you get the gist w/o suffering thru the math

Imagine an expanding (or contracting) sphere of uniform density (homogeneous

$$M_s = \rho(t)\, V(t) = \frac{4\pi}{3}\rho(t)\, R_s(t)^3$$

$$F = -\frac{G M_s m}{R_s(t)^2} = ma \; ; \quad a = \frac{d^2}{dt^2} x_m = \frac{d^2}{d}$$

multiply $\frac{dR}{dt}$ on each side:

$$\frac{dR}{dt}\frac{d^2R}{dt^2} = -\frac{GM_s}{R_s(t)^2}\frac{dR}{dt}$$

$$\boxed{\frac{d}{dt}\dot{r}^2 = 2\dot{r}\frac{d}{dt}\dot{r} = 2\dot{r}\ddot{r}} \; ; \quad \frac{d}{dt}\frac{1}{R} = -\frac{1}{R^2}\frac{dR}{dt}$$

$$\text{so} \quad \underbrace{\underbrace{\frac{1}{2}\left(\frac{dR}{dt}\right)^2}_{\text{kinetic } E} = \underbrace{\frac{GM_s}{R(t)}}_{\text{potential } E}}_{\text{per unit mass}} + U$$

Let's scale the radius of the sphere in dimensionless units : $R_s(t) = a(t)\, r_s$

comoving radius - what is tha

$$\frac{dR_s}{dt} = \frac{d}{dt}\left(a(t)\, r_s\right) = r_s\, \dot{a}$$

$$\Rightarrow \quad \frac{1}{2} r_s^2 \dot{a}^2 = \frac{G\, 4\pi}{3} \rho\, R_s^3 / R_s + U = \frac{4\pi G}{3} \rho\, r_s^2 a^2 + U$$

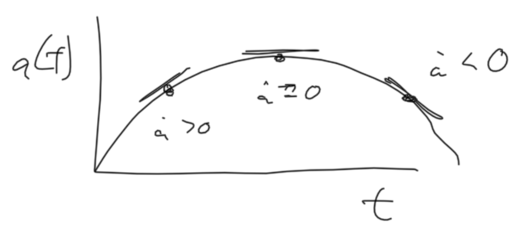
rearrange :

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho(t) + \frac{2U}{r_s^2} \frac{1}{a^2}}$$

Since $\dot{a}$ is squared, equation same whether sphere $\leftarrow\rightarrow$ or $\rightarrow\leftarrow$; take expanding case

$U > 0$: $\dot{a}^2$ always $+$, so always expanding

$U < 0$: ρ starts out v. high, so will be positive, but $\rho \downarrow$ w/time so eventually $\dot{a} = 0$: stops expand & becomes negative $\rightarrow$ contracting



can skip

$$\frac{8\pi G}{3}\rho = \frac{\cancel{8\pi} G}{\cancel{3}} \frac{M_s}{\frac{4\pi}{3} r_s^{\cancel{3}} a^{\cancel{3}}} = -\frac{2U}{r_s^2}\frac{1}{a^2}$$

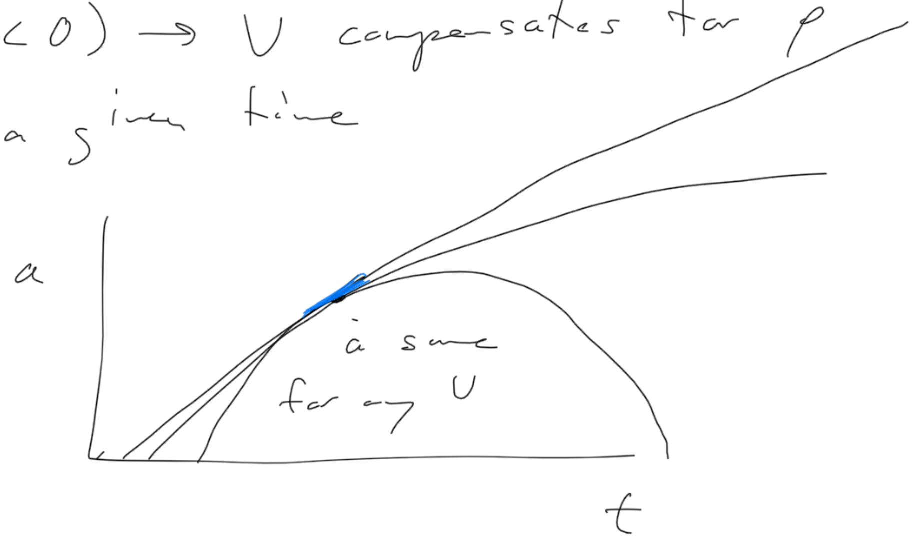
$$a = -\frac{GM_s}{U r_s} \quad (\text{where } U<0)$$

$U=0$: $$\left(\frac{\dot a}{a}\right)^2 = \frac{2GM_s}{r_s^3 a^3} \rightarrow \boxed{\dot a^2 \propto a^{-1}}$$

as $a\uparrow$, $\dot a\downarrow$ until $\dot a = 0$ @ $a\to\infty$

→ solve for ρ, get "critical density" (given $H_0 = \left(\frac{\dot a}{a}\right)_a$)

If $\rho < \rho_{crit}$, sphere expands forever ($U>0$), if $\rho > \rho_{crit}$ then collapses ($U<0$) → U compensates for ρ @ a given time



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* Works for ∞ ρ dist. too, since Newtonian only cares about shells interior, not exterior

Relativistic equation similar

Intro GR 1st

$$\boxed{\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{Kc^2}{R_0^2}\frac{1}{a(t)^2}}$$

start here on thurs

$$\rho \to \frac{\varepsilon}{c^2}, \quad \frac{2U}{r_s^2} \to -\frac{Kc^2}{R_0^2}$$

$$\boxed{E = \gamma m c^2}, \quad \boxed{p = \gamma m v}, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

get rid of

$$p^2 = \frac{m^2 v^2}{1 - v^2/c^2}$$

$$p^2 = m^2 v^2 + p^2 v^2/c^2$$

$$v = \frac{p}{\sqrt{m^2 + p^2/c^2}}$$

$$p^2 = \frac{m^2 p^2}{(m^2 + p^2/c^2)}\gamma^2$$

$$\gamma^2 = \frac{p^2(m^2 + p^2/c^2)}{p^2 m^2} = 1 + \frac{p^2}{m^2c^2}$$

$$\boxed{E = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}} = \sqrt{m^2c^4 + p^2c^2}}$$

If $v \ll c$, $\gamma \to 1$ & $p \simeq mv$ so

$$E \approx \sqrt{m^2c^4 + m^2v^2c^2} = mc^2\sqrt{1 + v^2/c^2}$$

expand $(1 + v^2/c^2)^{1/2} \to 1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots$

$$E = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$$

$$\boxed{E = mc^2 + \frac{1}{2}mv^2}$$

particle rest E — will dominate for slow, massive particles

kinetic E

BUT, $v \approx c$ particles, like light, have

$$E_{rel} = pc = h\nu \quad (= hf)$$

☆ Both particles & radiation contribute to gravity, hence $\varepsilon(t)$ instead of $\rho(t)$

Recall $v = H_0 d \to v(t) = H(t)\,d(t)$

& $H(t) = \frac{\dot{a}}{a}$

$$H(t)^2 = \frac{8\pi G}{3c^2}\,\varepsilon(t) - \frac{Kc^2}{R_0^2\,a(t)}$$

Today, have: $$H_0^2 = \frac{8\pi G}{3c^2}\,\varepsilon_0 - \frac{Kc^2}{R_0^2}$$

Boundary case is $K=0$, so the critical (energy) density is

$$\boxed{\varepsilon_{crit,0} = \frac{3c^2 H_0^2}{8\pi G}}$$

In a matter-dominated time (like now, if no dark energy), can also write as a mass density $\rho_{c,0} = \varepsilon_{c,0}/c^2$

$$\boxed{\begin{aligned}\varepsilon_{crit,0} &\sim 5000\ \mathrm{MeV/m^{-3}}\\ \rho_{crit,0} &\sim 10^{11}\ M_\odot/\mathrm{Mpc}^3\end{aligned}}$$

mass proton/neutron ~ 1000 MeV or $5/m^3$

MW $\sim 10^{12} M_\odot$, so 1 large gal. per 3×3 M box

B/c the numbers are weird, more useful to define energy densities @ ratios to critical values

$$\Omega(t) \equiv \frac{\varepsilon(t)}{\varepsilon_c(t)}, \quad \Omega_0 = \frac{\varepsilon(t_0)}{\varepsilon_c(t_0)}$$

Sub into Friedmann eq.

$$H_0^2 = \frac{8\pi G}{3c^2}\varepsilon_{c,0} = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{Kc^2}{R_0^2}$$

$$1 = \Omega_0 - \frac{3c^4 K}{8\pi G R_0^2}$$

$$\boxed{1 - \Omega_0 = -\frac{Kc^2}{R_0^2 H_0^2}}$$

or

$$\boxed{1 - \Omega(t) = -\frac{Kc^2}{R_0^2 H(t)^2 a(t)^2}}$$

$$\frac{K}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1) = \frac{\Omega_0 - 1}{d_H^2}$$

$$\Omega_0 \rightarrow K$$

$$d_H \rightarrow R_0$$

Amazing as it is, STILL don't know how to solve for $a(t)$

→ Newtonian Friedmann eq. derived essentially via E. conservation $U_{grav} + K = \text{const.}$

→ 1ST law of thermodynamics is also an E. conservation statement

$$dQ = dE + PdV$$

flow in/out heat, internal E

Homogeneity implies $dQ = 0$, so evolution w/time: $\dot{E} + P\dot{V} = 0$

again $V(t) = \frac{4\pi}{3} r_s^3 a(t)^3$

$$\dot{V} = 4\pi r_s^3 a^2 \dot{a} = \boxed{V\left(3\frac{\dot{a}}{a}\right) = \dot{V}}$$

internal E is just the density × volume

$$E(t) = V(t)\,\varepsilon(t)$$

$$\dot{E} = \varepsilon\dot{V} + V\dot{\varepsilon} = \varepsilon V 3\frac{\dot{a}}{a} + V\dot{\varepsilon}$$

$$\boxed{\dot{E} = V\left(\dot{\varepsilon} + 3\varepsilon\frac{\dot{a}}{a}\right)}$$

so $\dot{E} + P\dot{V} = 0$ becomes

$$\cancel{V}\left(\dot{\varepsilon} + 3\frac{\dot{a}}{a}\varepsilon\right) + P\cancel{V}3\frac{\dot{a}}{a} = 0$$

$$\boxed{\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0}$$

Fluid Equation

→ now we can modify the F.E. to get the Acceleration Equation

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\varepsilon a^2 - \frac{Kc^2}{R_0^2}$$

derivative: $$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\left(2a\dot{a}\varepsilon + a^2\dot{\varepsilon}\right)$$

$\dot{\varepsilon} = -3\frac{\dot{a}}{a}(\varepsilon + P)$, so sub. & rearranging

get rid of $\dot{a}$

$$\ddot{a} = \frac{1}{2\cancel{\dot{a}}}\frac{\overset{4}{\cancel{8}}\pi G}{3c^2}\left(2a\cancel{\dot{a}}\varepsilon + a^2\left[-3\frac{\cancel{\dot{a}}}{a}(\varepsilon + P)\right]\right)$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(2\varepsilon - 3[\varepsilon + P]\right)$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}[\varepsilon + 3P]}$$

$\varepsilon \rightarrow$ always positive, pressure of particles & radiation also positive, so $\ddot{a} < 0$ in principle

If something had negative pressure, such that $p < -\frac{1}{3}\varepsilon$, then you would get accelerated expansion (so if you measure that, universe must be dominated by something weird)

Have F.E. $[a(t), \varepsilon(t), R_0]$

Fluid Eq. $[\varepsilon(t), a(t), P]$

Acc. Eq. $[a(t), \varepsilon(t), P]$

2 inde[...] eq-, 3 vari[...]

Need 3rd Eq: "equation of state"

$$\boxed{P = P(\varepsilon)}$$

For a "dust-filled" universe, state acts like a dilute gas so

$$\underline{P \propto \varepsilon} \rightarrow \boxed{P = w\varepsilon}$$

Can assume the ideal gas law: $PV = NRT$

or $P = \frac{\rho}{\mu} kT$

For non-rel. gas, $E = mc^2 + \frac{1}{2}mv^2 \approx mc^2$

so $\varepsilon = \rho c^2$ thus $P \approx \frac{kT}{\mu c^2}\varepsilon$

Gas has a Maxwellian dist., which obeys $3kT = m\langle v^2 \rangle$

so $P \simeq \frac{kT}{mc^2}\varepsilon = \underbrace{\frac{\langle v^2 \rangle}{3c^2}}_{w}\varepsilon$

nonrel. $\rightarrow$ $v^2 \ll c^2$, so $w \ll 1$

(room temp., $w \sim 10^{-12}$)

so for rel. particles $\langle v^2 \rangle \sim c^2$, so

$P_{rel} = \frac{1}{3}\varepsilon_{rel} \rightarrow w = \frac{1}{3}$

Revisit Acc. Eq., $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)$

matter: $P \to 0$, decel.

rad.: $\varepsilon + 3P = 2\varepsilon$, decel.

$\boxed{w < -\frac{1}{3}}$: $\varepsilon + 3P < 0$, $\frac{\ddot{a}}{a} > 0 \rightarrow$ accel.!

dark energy

$\Lambda \rightarrow \underline{\underline{w = -1}}$ so $P = -\varepsilon$

$\hookrightarrow$ measuring this is major task in cosmology today

Add Λ to Einstein's Field Equations

get $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3}$

Fluid Eq. unchanged, & accel. eq. becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) + \frac{\Lambda}{3}$$

Static Universe $\rightarrow$ $\ddot{a} = 0$ & $\underbrace{\varepsilon}_{\text{rest mass}} + 3\underbrace{P}_{\simeq 0} \rightarrow \rho c^2$

$$\boxed{\Lambda = 4\pi G\rho}$$

$\dot{a} = 0$ also, so $\frac{\kappa c^2}{R_0^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$

+ so $\kappa = +1$, can solve for R_0

$$R_0 = \frac{c}{2\sqrt{\pi G\rho}} = \frac{c}{\Lambda^{1/2}}$$

But this is not our universe, except Λ <u>is</u> real $\rightarrow$ but where does it come from

If vacuum energy (virtual particles), then there E via the uncertainty principle: $\Delta E \Delta t \leq h$

But what? $\varepsilon_{vac} \sim \frac{E_P}{l_P^3} \sim 10^{132}\ \mathrm{eV\,m^{-3}}$ $\Big\}$ 123 ord of mag.!

measured value is $\sim 10^9\ \mathrm{eV\,m^{-3}}$