ASTR 4080 - Week 4

Unknowns : $\mathcal{E}(t)$, $a(t)$, $P(\epsilon)$

↳ also K/R_0 : fundamental constants, possibly determined by contents

Friedmann Eq:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\mathcal{E} - \frac{Kc^2}{R_0^2 a^2}$$

Fluid Eq:

$$\dot{\mathcal{E}} + 3\,\frac{\dot{a}}{a}\left(\mathcal{E} + P\right) = 0$$

Eq. of State: $P = w\,\mathcal{E}$

(put on left side board)

Slide 2 Whatever is in universe, the E. densities simply sum

* Fluid Eq. holds for each component separately then

On right board

$$\dot{\varepsilon}_i + 3\frac{\dot{a}}{a}(\varepsilon_i + P) = 0$$

$$\dot{\varepsilon}_i + 3\frac{\dot{a}}{a}\varepsilon_i(1+w_i) = 0$$

$$\frac{1}{\varepsilon_i}\,d\varepsilon_i = -\frac{3}{a}(1+w_i)\,da$$

Assuming $w_i \to$ const., integrate

$$\ln \varepsilon_i = -3(1+w_i)\ln a + C$$

$$\boxed{\varepsilon_i(a) = \varepsilon_{i,0}\, a^{-3(1+w_i)}}$$

Already know w for non-rel.,
rel., & Λ components

$\underline{w=0}$ $\underline{w=\frac{1}{3}}$ $\underline{w=-1}$

$\varepsilon_m = \varepsilon_{m,0}/a^3$ $\varepsilon_r = \varepsilon_{r,0}/a^4$ $\varepsilon_\Lambda = \varepsilon_{\Lambda,0}$

☆ refer to expansion on slide

→ Next Slide

On right board

$$\frac{\varepsilon_\Lambda(a)}{\varepsilon_m(a)} = \frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}\, a^{-3}} = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\, a^3$$

$a = 0.77$ @ $\varepsilon_\Lambda = \varepsilon_m$

$$\frac{\varepsilon_m(a)}{\varepsilon_r(a)} = \frac{\varepsilon_{m,0}\, a^{-3}}{\varepsilon_{r,0}\, a^{-4}} = \frac{\Omega_{m,0}}{\Omega_{r,0}}\, a = \frac{1}{3400}$$

$1+z = \frac{1}{a}$, so $z_{m\Lambda} = 0.31$ & $z_{rm} \cong 3400$
$z_{gal} \sim 10$, $z_{CMB} \sim 1000$

Convenient to write Friedmann Eq.

as $$\boxed{\dot{a}^2 = \frac{8\pi G}{3c^2} \sum_i \varepsilon_{i,0}\, a^{-1-3w_i} - \frac{Kc^2}{R_0^2}}$$

☆ write on left board

w/ this formulation, can solve for $a(t)$

for different kinds of universes

☆ Briefly to Summary slide then Empty slide

then back to board

$$\dot{a}^2 = -\frac{Kc^2}{R_0^2} \rightarrow K = -1$$

(+1 forbidden)

- if spacetime flat, no expansion just static

otherwise, $\dot{a} = \pm \frac{c}{R_0}$ & $\boxed{a(t) = t/t_0}$

[HW problem w/ $\Gamma = 1$]

- Observe light from distant galaxy, have

$$1+z = \frac{1}{a(t_e)} = \frac{t_0}{t_e}$$

$$\& \ t_e = \frac{H_0^{-1}}{1+z}$$

What distance is the galaxy at?

$$d_p(t_0) = a(t_0)\int_0^r dr = r$$

↳ @ current time

Light follows null geodesic: $ds^2 = -c^2dt^2 + a(t)^2 d\ldots = 0$

$$c\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^r dr = r$$

Thus $$\boxed{d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)}}$$ ☆ Generally true

Empty: simple → $$\boxed{d_p(t_0) = \frac{c}{H_0}\ln(1+z)}$$

- Can see farther than Hubble sphere! but not distance when photon emitted

$$\boxed{d_p(t_e) = \frac{1}{1+z} d_p(t_0)}$$

Horizon distance → most distant obj. can see

equiv. to $d_p(t_0)$ as $z \to \infty$ → $$d_{hor}(t_0) \equiv c\int_0^{t_0} \frac{dt}{a(t)} = ct_0\int_0^{t_0} \frac{dt}{t} = ct_0 [\ln t_0 - \ldots] = \infty$$

☆ of course, nothing to see if empty...

☆ stay on Empty slide, save solutions on board boxed in yellow

$K=0 \rightarrow$ $\boxed{\dot{a}^2 = \frac{8\pi G \varepsilon_0}{3c^2} a^{-(1+3w)}}$

< math happens >

$\boxed{a(t) = \left(\frac{t}{t_0}\right)^{\frac{2/3}{1+w}}}$, $\boxed{t_0 = \frac{1}{1+w}\frac{c}{\sqrt{6\pi G \varepsilon}}}$

if $w \neq -1$

$\dot{a}^2 = \frac{8\pi G \varepsilon_0}{3c^2} a^2 \rightarrow \dot{a} = H_0 a$

$\boxed{a(t) = e^{H_0(t-t_0)}}$ flat Λ-only

For matter, $w \approx 0$ and $\boxed{a(t) = (t/t_0)^{2/3}}$

Radiation, $w = \frac{1}{3}$ & $\boxed{a(t) = (t/t_0)^{1/2}}$

☆ Next Slide

$$\varepsilon_{crit,0} = \frac{3c^2}{8\pi G} H_0^2$$

$$H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0} = \frac{2}{3}\frac{1}{(1+w)} t_0^{-1}$$

so $$\varepsilon_{crit,0} = \frac{c^2 t_0^{-2}}{6\pi (1+w)^2}$$

Matter : Einstein-de Sitter Universe

$$\hookrightarrow t_0 = \frac{2}{3}\frac{1}{H_0}$$

$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{(t/t_0)^{2/3}} = 3ct_0\left[1-\left(\frac{t_e}{t_0}\right)^{1/3}\right]$$

$$= \frac{2c}{H_0}\left[1-\frac{1}{\sqrt{1+z}}\right]$$

Radiation : $t_0 = \frac{1}{2}\frac{1}{H_0}$

$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{(t/t_0)^{1/2}} = 2ct_0\left[1-\left(t_e/t_0\right)^{1/2}\right]$$

$$= \frac{c}{H_0}\frac{z}{(1+z)}$$

Lambda : de Sitter Universe

$$t_0 \to \infty \; ! \quad a(t=0) = e^{-H_0 t_0} \equiv 0$$

$$d_p(t_0) = c\int_{t_e}^{t_0} e^{H_0(t_0-t)}\,dt = \frac{c}{H_0}\left[e^{H_0(t_0-1)}-1\right]$$

$$= \frac{c}{H_0} z$$

★ Next Slide → discuss then come back here

Multi-component Universes

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a^2}$$

$$\varepsilon(t) = \sum_i \varepsilon_i(t)$$

In terms of critical density, becomes

$$\frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1) \quad @ \ t = t_0$$

$$\varepsilon_{crit,0} \equiv \frac{3c^2 H_0^2}{8\pi G}$$

sub these 2 into to get

so Friedmann Eq. becomes:

$$\frac{H(t)^2}{H_0^2} = \frac{\varepsilon(t)}{\varepsilon_{crit,0}} + \frac{1-\Omega_0}{a(t)^2}$$

$$\Omega_i \equiv \frac{\varepsilon_i(t)}{\varepsilon_{crit,0}}, \quad \varepsilon_i(a) = \varepsilon_{i,0}\, a^{-3(1+w_i)}$$

so $$\frac{H^2}{H_0^2} = \frac{\dot{a}^2}{a^2}\frac{1}{H_0^2} = \sum \Omega_i + \frac{1-\Omega_0}{a^2}$$

$$\Omega_0 = \sum \Omega_{i,0} \ \& \ \Omega_i = \Omega_{i,0}\, a^{-3(1+w_i)}$$

so $$H_0^{-1}\dot{a} = \left[\Omega_{r,0}\, a^{-2} + \Omega_{m,0}\, a^{-1} + \Omega_{\Lambda,0}\, a^{2} + (1-\Omega_0)\right]^{1/2}$$

$$= E(a)$$

And $$\int_0^a \frac{da}{Ea} = \int_0^t H_0\, dt$$

$$\boxed{t = \frac{1}{H_0}\int_0^a \frac{da}{E(a)}}$$

No analytic solution in the general case, but certain limits are solvable ($\Omega_{i,0} = 0$)

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$\boxed{\text{Matter + Curvature}}$ $K=0$ already solved

$K = +1$, $a_{max} = \frac{\Omega_0}{\Omega_0 - 1} \rightarrow \underline{\Omega_0 > 1}$

Friedmann Eq. is $\frac{\dot{a}^2}{H_0^2} = \frac{\Omega_0}{a} + (1-\Omega_0)$

curvature gone via $1-\Omega_0 = -\frac{Kc^2}{R_0^2 H_0^2}$

↳ determined by $\underline{\Omega_0}$

$$t = \frac{1}{H_0}\int_0^a \frac{da}{E(a)} = \frac{1}{H_0}\int_0^a \frac{da}{\sqrt{\Omega_0 a^{-1} + (1-\Omega_0)}}$$

Can solve for $\Omega_0 \neq 1$ case (already solved

☆ Next Slide

Ignoring Radiation (Matter/Λ/Curvature

$\rightarrow$ valid @ late times

$K=0$, $\Omega_{m,0}+\Omega_{\Lambda,0}=1$

F. Eq. $$\frac{H^2}{H_0^2}=\frac{\Omega_{m,0}}{a^3}+(1-\Omega_{m,0})$$

always ⊕ also ⊕ if $\Omega_{m,0}$

Will always expand (if currently expanding)

If $\Omega_{m,0}>1$, $\Lambda<0$ & provides an attraction force $\rightarrow$ get a "Big Crunch"

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In the $\Omega_{\Lambda,0} > 0$ case, can integrate for an analytic solution

$$H_0 t = \frac{2}{3\sqrt{1-\Omega_{m,0}}} \ln\left[\left(\frac{a}{a_{m\Lambda}}\right)^{3/2} + \sqrt{1+\left(\frac{a}{a_{m\Lambda}}\right)^3}\right]$$

where $a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{1/3}$

~~@~~ the time when $\underline{\Omega_m = \Omega_\Lambda}$

At early times, b/c $\varepsilon_m \propto a^{-3}$ & $\varepsilon_\Lambda = \text{const.}$ matter dominates when $a \ll a_{m\Lambda}$

$$\ln[\,] \to \ln\left[1+\left(\frac{a}{a_{m\Lambda}}\right)^{3/2}\right] \simeq \left(\frac{a}{a_{m\Lambda}}\right)^{3/2}$$

so $$H_0 t \simeq \frac{2}{3\sqrt{1-\Omega_{m,0}}} \frac{a^{3/2}}{\left(\left[\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right]^{1/3}\right)^{3/2}}$$

$$\Rightarrow a(t) \simeq \left(\frac{3}{2}\Omega_{m,0}^{1/2} H_0 t\right)^{2/3} \propto t^{2/3}$$

as expected for a flat, matter-dominated universe

When $a \gg a_{m\Lambda}$ (Λ dominates)

$$\ln[\;] \approx \ln\left[2\left(\frac{a}{a_{m\Lambda}}\right)^{3/2}\right] = \ln 2 + \frac{3}{2}\ln$$

$$\approx \frac{3}{2}\ln\frac{a}{a_{m\Lambda}}$$

so $$e^{H_0 t\sqrt{1-\Omega_{m,0}}} \approx \frac{a}{a_{m\Lambda}}$$

or $$a(t) = a_{m\Lambda}\, e^{H_0\sqrt{1-\Omega_{m,0}}\, t}$$

$\underline{a(t) \propto e^{Kt}}$ $\rightarrow$ flat, Λ-dominat

Equation also gives t_0 (for $a_0 = 1$)

$$t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}}\ln\left[\frac{\sqrt{1-\Omega_{m,0}}+1}{\Omega_{m,0}^{1/2}}\right]$$

$\rightarrow$ $\Omega_{m,0} \sim 0.3$, $\Omega_{\Lambda,0} \sim 0.7$ so

$$t_0 \sim 0.96\, H_0^{-1} = 0.96\, t_H$$

$H_0 = 70$: $t_H = 14\,\text{Gyr}$ & $t_0 = 13.5\,\text{Gyr}$

$H_0 = 68$: $t_H = 14.5\,\text{Gyr}$ & $t_0 = 13.9\,\text{Gyr}$

$\Omega_{m,0} = 0.31$ / $\Omega_{\Lambda,0} = 0.69$ $\rightarrow$ $\boxed{t_0 = 13.76}$

Add Curvature $\rightarrow$ no analytic solution

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Radiation + Matter

Early times, ε_Λ insignificant, but $\varepsilon_r \propto a^{-4}$

faster than ε_m so do same exercise

for rad + matter

Flat: $$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}$$

$H = \frac{\dot{a}}{a}$, rearrange to integrate

$$H_0 dt = \frac{a\,da}{\Omega_{r,0}^{1/2}} \frac{1}{\sqrt{1 + a/a_{rm}}}$$

$$a_{rm} = \frac{\Omega_{r,0}}{\Omega_{m,0}}$$ $\hookrightarrow$ <u>integrate</u>

$$H_0 t = \frac{4 a_{rm}^2}{3\Omega_{r,0}^{1/2}} \left[1 - \left(1 - \frac{a}{2a_{rm}}\right)\left(1 + \frac{a}{a_{rm}}\right)^{1/2} \right]$$

can skip

So does the equation behave correctly when are dominates?

Matter : $a \gg a_{rm}$

$$H_0 t \propto \left[1-\left(-\frac{a}{2a_{rm}}\right)\left(\frac{a}{a_{rm}}\right)^{1/2}\right]$$

$$\propto a^{3/2}$$

thus $a \propto t^{2/3}$ → same as before

Radiation : $a \ll a_{rm}$

$$\left(1+\frac{a}{a_{rm}}\right)^{1/2} \approx 1+\frac{1}{2}\frac{a}{a_{rm}}$$

$$\left(1-\frac{a}{2a_{rm}}\right)\left(1+\frac{a}{2a_{rm}}\right) = 1+\left(\frac{a}{2a_{rm}}\right)^2$$

$$H_0 t \propto a^2 \rightarrow a \propto t^{1/2}$$

Can solve for time radiation = matter

$t_{rm} = 50\,kyr$

So, radiation epoch short → for long term evolution of a, can be ignored

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