Homework 6

Due **April 2 at 2pm via email**
Please show all work, writing solutions/explanations clearly, or no credit will be given. You are encouraged to work together, but everyone must turn in independent solutions: do not copy from others or from any other sources.

1. A magnetar is discovered near the edge of a supernova remnant with a radius of $R_{\rm SNR} = 2$ pc. Proper motion observation indicate that the magnetar is moving radially outward at a speed of $v = 1000$ km/s.

   (a) Assuming the magnetar was born near the center of the SNR, estimate its age in years.

   (b) The steady X-ray output and gamma-ray output during SGR bursts show a pulsation period of $P = 8$ seconds. Assume that this is the rotation period of the neutron star. Assume the magnetar is slowing down by the magnetic dipole formula, and that its magnetic moment μ, moment of inertia $I = 10^{45}$ gm cm^2, and magnetic inclination $\alpha = 90^\circ$ are all constant, Assume the neutron star was born with a much shorter period, $P_o \ll P$. Estimate the slowing down rate $\dot{P}$.

   (c) Making the same assumptions as above, estimate the magnetic dipole moment μ of the magnetar (in G cm^3).

   (d) Assuming the radius of the neutron star is 10 km, estimate the surface polar field B_p of the neutron star (in G). (A version of this argument was used to predict the magnetic fields in SGRs prior to the actually measurement of $\dot{P}$ from X-ray pulsations.)

2. Adapt the outline you made for your project to the template for the written report, putting text into all sections as much as you currently can. Write a complete draft of your abstract, making up your expected results or conclusions in line with your expectations. Submit a PDF copy of this draft along with the solutions to the above questions.

CONTINUED ON NEXT PAGE

3. In both Newtonian mechanics and General Relativity (GR), it is useful to describe test particle motion in a force field by an "effective potential" $V_{\rm eff}(r)$. In Newtonian mechanics for a spherically symmetric force field, the energy is given by $E = (1/2)mv_r^2 + V_{\rm eff} \geq V_{\rm eff}$, where m is the test particle mass, and $v_r = dr/dt$ is the radial component of the velocity. Since $v_r^2 \geq 0$, the particle motion is limited to radii where $E \geq V_{\rm eff}(r)$. For a central point mass M, the Newtonian effective potential is $V_{\rm eff} = -GM/r + L^2/(2mr^2)$, where L is the orbital angular momentum of the test mass.

   For the Schwarzschild metric in GR, the corresponding equation for the energy is $E = \sqrt{m^2c^2v_r^2 + V_{\rm eff}^2} \geq V_{\rm eff}$. Here, the radial velocity is $v_r = dr/d\tau$, and the effective potential is

$$V_{\rm eff} = mc^2 \sqrt{\left(1 - \frac{2GM}{c^2 r}\right)\left(1 + \frac{L^2}{m^2c^2r^2}\right)} \, .$$

   The equation for the radius r of a circular orbit, derived from this potential, is

$$r = \frac{6GM}{c^2}\left(1 - \sqrt{1 - \frac{12G^2M^2m^2}{c^2L^2}}\right)^{-1} \, .$$

   (a) For the Schwarzschild metric, what is the limiting value of $V_{\rm eff}$ as $r \to \infty$? Does this make sense?

   (b) Orbits with very large radii also have very large orbital angular momenta. What is the radius of a circular orbit for large values of L?

   (c) Show that the result of part (b) agrees with Kepler's Third Law, so that the Newtonian limit is correct.

   (d) What is the orbital angular momentum $L_{\rm lso}$ of the last stable circular orbit (the stable circular orbit with the smallest radius r and angular momentum L)?

   (e) Show that the radius of the last stable circular orbit is $R_{\rm lso} = 6GM/c^2 = 3R_{\rm sch}$, where $R_{\rm sch}$ is the Schwarzschild radius.

   (f) What is the energy E of a particle of mass m in the last stable circular orbit?

   (g) If the particle in part (f) started at large radii and moved into the last stable circular orbit through an accretion disk, how much energy did it lose (presumably in radiation) in moving into this point?