ASTR 5590 - Chapter 5

Massive particle speeding along, collides w/ less massive

non-relativistic

$\overset{\bullet \rightarrow}{M} \; v \qquad \overset{\bullet}{m}$

Convert to CoM frame, head-on collision

$\overset{\rightarrow v_M}{\underset{M}{\bullet}} \quad | \quad \overset{v_m \leftarrow}{\underset{m}{\bullet}}$

$$V_m = \frac{-M}{M+m} v$$

$$V_M = \frac{m}{M+m} v$$

$v_m \leftarrow \bullet | \rightarrow v_m$

$$\Delta v_m = \frac{2M}{M+m} v, \text{ if } m \ll M, \; \Delta v_m \simeq 2v$$

What is KE lost by M?

$$\Delta KE = -\tfrac{1}{2} m (2v)^2 = -2mv^2$$

$$\frac{\Delta KE}{KE} = -\frac{2mv^2}{\frac{1}{2} M v^2} = -4 \frac{m}{M} \begin{matrix} \leftarrow e^- \\ \leftarrow p^+ \end{matrix}$$

what is %?

$$\frac{4}{1836} \Rightarrow \boxed{0.2\%}$$

→ ion is undeviated, loses some E but carries on whilst e^- is sent packing

How much E is lost during a collision?

Show 5.1 + 5.2

do these together w/ classes

Describe 5.1 ~~assumptions~~ / definitions

→ want to know E transferred to e^- during collision, how can we do that?

Figure out amount of momentum transferred to e^- during the interaction

Recall $\vec{F} = \frac{d\vec{p}}{dt}$, so $\Delta p = \int \vec{F} dt$

How can we simplify encounter? symmetries?

What is force? $\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}$, $t = \frac{x}{v}$ (const.)

$F_{\parallel}$ cancels out, $q_1 = ze$, $q_2 = -e$ (e^-)

$$F = \frac{ze^2}{\quad} \sin\theta \qquad dt = \frac{dx}{v}$$

$F_\perp \quad 4\pi\epsilon_0 r$

From diagram, $\sin\theta = \frac{b}{r}$ & $\tan\theta = \frac{b}{x}$

assume encounter quick, so $b \sim$ const

Have θ, r, x as variables, convert to θ

$$dx = \frac{d}{d\theta}\left(\frac{b}{\tan\theta}\right) = b\frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta}\right) = b\,\frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}d\theta$$

$$= -\frac{b}{\sin^2\theta}d\theta$$

$$dt = -\frac{b}{v}\frac{d\theta}{\sin^2\theta}, \quad F_\perp = \frac{ze^2}{4\pi\epsilon_0 b^2}\sin^3\theta$$

$$\Delta p = \int_{-\infty}^{\infty} F_\perp(x)\,dt(x) = -\int_0^{\pi} F_\perp(\theta)\,dt(\theta)$$

$$= -\frac{ze^2}{4\pi\epsilon_0 bv}\underbrace{\int_0^{\pi}\sin\theta\,d\theta}_{-2}$$

$$= \frac{ze^2}{2\pi\epsilon_0 bv}$$ ☆ keep

Really want to know $\Delta E = \Delta KE = \frac{p^2}{2m_e}$

$$= \frac{z^2e^4}{8\pi^2\epsilon_0^2 b^2 v^2 m_e}$$

This is for 1 e^- @ impact param. b

In a material w/ density N_e, need to integrate over all b's w/in an interval dx to find lost E w/ distance

$-dE$ in length dx comprises integral of encounters w/ e^-'s in the volume

Fig. 5.2 $\rightarrow$ $\underbrace{2\pi b}\,\underbrace{db}\,\underbrace{dx}$

$$-dE = \frac{z^2e^4}{8\pi^2\epsilon_0^2 v^2 m_e}\underbrace{\int_{b_{min}}^{b_{max}}\frac{2\pi b\,db}{b^2}}_{\int\frac{1}{b}\rightarrow \ln b}dx$$

$$-\frac{dE}{dx} = \frac{z^2e^4}{4\pi\epsilon_0^2 v^2 m_e}\ln\frac{b_{max}}{b_{min}}$$

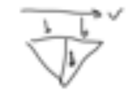
What are b_{max}/b_{min}? Fudge factors!

Classical calculation; correct derivations way beyond scope of class

What is b_{max}? If the collision lasts as long ^or longer as it takes the e^- to complete an orbit → no ionization occurs

What is timescale τ? $p = F\tau = \frac{ze^2}{2\pi\epsilon_0 b v}$

where F is Coulomb force @ $r=b$



$$F = \frac{ze^2}{4\pi\epsilon_0 b^2}, \quad \text{so} \quad \tau = \frac{2b}{v}$$

$$\tau \approx \frac{1}{\nu_0} = \frac{2b_{max}}{v}, \quad \boxed{b_{max} = \frac{\pi v}{\omega_0}}$$

where $\nu_0 = \frac{\omega_0}{2\pi}$

What about b_{min}?

classical + quantum options

↳ PE is equal to max KE transfer,
or $2m_e v^2 \approx \frac{ze^2}{4\pi\epsilon_0 b_{min}}$

$$\boxed{b_{min} = \frac{ze^2}{8\pi\epsilon_0 m_e v^2}} \text{ classical}$$

comparable to distance e^- moves during encounter $\tau = \frac{2b}{v}$ (modulo factor 8)

quantum: uncertainty principle, $\Delta v = 2v$ like before

so $\Delta p = m_e \Delta v = 2m_e v$

$$\Delta x \approx \hbar/\Delta p \approx \boxed{\hbar/2m_e v = b_{min}}$$

✱ use whichever of 2 choices is larger

$$\frac{b_{min}(q)}{b_{min}(cl)} = \frac{4\pi\epsilon_0 v\hbar}{ze^2} = \frac{1}{z\alpha}\left(\frac{v}{c}\right) = \frac{137}{z}\frac{v}{c}$$

↑ fine structure const.

need to use $b_{min}(q)$ if $v/c \gtrsim 0.01$
non-rel. regime still

↑v, $v/c > 0.01$, $-\frac{dE}{dx} = \frac{z^2 e^4 N_e}{4\pi\epsilon_0^2 v^2 m_e} \ln\left(\frac{2m_e v^2}{\hbar\omega_0}\right)$

Bohr model, ω_0 is ground state orbit
[illegible] binding E or Ionization pot. I

$$I = \tfrac{1}{2}\hbar\omega_0$$

Composition will be varied, so $\bar{I}$ more appropriate, & write as

$$-\frac{dE}{dx} = \frac{\sim}{\sim} \ln\left(\frac{\overbrace{m_e v^2}^{2KE}}{\bar{I}}\right)$$

Can get same result by working out E spectrum of ejected electrons (HW!)

Relativistic Case, $v \sim c$

Fig 5.3, $t = t' = 0$ & $x = x' = 0$ @ closest approach

Show equations on slide

use Lorentz transformations

Fig. 5.4

to explain difference in forces (time dilation)

Ionization losses, same picture

$$\int F_z\, dt = \int_{-\infty}^{\infty} e E_z\, dt$$

$$= \frac{z e^2 \gamma b}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dt}{\left[b^2 + (\gamma v t)^2\right]^{3/2}}$$

$$= \frac{z e^2 \gamma b}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dt}{b^3\left[1 + \left(\frac{\gamma v t}{b}\right)^2\right]^{3/2}}$$

$$= \frac{z e^2}{2\pi\epsilon_0 v b}$$

same as nonrel. case! $\Delta p \sim \underset{\propto\gamma}{\Delta E_z}\ \underset{\propto\frac{1}{\gamma}}{\tau}$

so γ factors cancel

Derivation of dE/dx same, except for b_{max} & $b_{min} \to \ln\left(\frac{E_{max}}{\bar{I}}\right)$

* How might this term change if relativistic?

$E_{max} \propto \gamma$ $(= \gamma m c^2)$

Similarly, $b_{min} = \frac{\hbar}{\Delta p}$ where $p = \gamma m v$ so

$\propto \frac{1}{\gamma}$

& $\frac{b_{max}}{b_{min}} \propto \gamma^2$, $= \frac{2\gamma^2 m_e v^2}{\bar{I}}$

Proper relativistic quantum calculation is

$$-\frac{dE}{dx} = \frac{z^2 e^4 N_e}{4\pi \epsilon_0^2 m_e v^2}\left[\ln\left(\frac{2\gamma^2 m_e v^2}{\bar{I}}\right) - \frac{v^2}{c^2}\right]$$

Bethe-Bloch formula

Fig. 55

Losses ↑ as E decreases, causing most KE to be lost some distance into a material