ASTR 5590 - White Dwarfs & Neutron Stars: Structure

WDs + NSs are supported by degeneracy pressure → basically uncertainty principle

↳ recall problem from the 1st day

Particles have same T, given by

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} kT$$

so $$p = mv \approx (3mkT)^{1/2}$$

Degeneracy pressure happens when matter is compressed, so Δx is minimized

$$\Delta x \approx \hbar / \Delta p$$

Take $\Delta p = p$, $$\rho = \frac{m_p}{(\Delta x)^3} \approx m_p \left(\frac{3mkT}{\hbar} \right)^{3/2}$$

b/c they dominate particle mass

Non-rel. limit, $\rho \propto T^{3/2}$

what we want is pressure & to find the equation of state (can't assume ideal gas)

In general, $P = (\gamma - 1)\varepsilon$ & $P \propto \rho^\gamma$

$$E_e = \frac{1}{2} m_e v^2 = \frac{p^2}{2m_e} \approx \frac{\hbar^2}{2m_e a^2} \quad \Delta x$$

so $\mathcal{E} \approx E/a^3 = \frac{\hbar^2}{2m_e a^5} \propto a^{-5}$

& $P \propto a^{-5}$ while $\rho \propto a^{-3}$

so $P \propto \rho^{5/3} \rightarrow \gamma = 5/3$

Turns out <u>non-rel.</u> degenerate matter has same γ as ideal gas

$$P \approx \frac{\hbar^2}{3m_e a^5} = \frac{\hbar^2}{3m_e}\left(\frac{\rho}{m_e}\right)^{5/3}$$

Want to generalize to other compositions

$$\rho = (n+n_e)\mu m_u = n\mu_o m_u = n_e \mu_e m_u$$

$m_u \rightarrow$ atomic mass unit $= 1.66\times10^{-27}$ kg

$1/12$ mass of atom of ^{12}C

$n_e \rightarrow$ # dens. e^-

$n \rightarrow$ # dens. of nuclei

$\mu m_u \rightarrow$ avg. mass per particle

$\mu_o m_u \rightarrow$ " " " nucleus

$\mu_e m_u \rightarrow$ " " " e^-

E.o.S. becomes $\boxed{P = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left(\frac{\rho}{\mu_e m_u}\right)^{5/3}}$

Can rewrite using $P = \frac{\rho kT}{\mu m_u}$ to find

$$\rho_{cr} = 2.4\times10^{-5}\left(\frac{1}{\mu}\ \frac{T}{K}\right)^{3/2}\mu_e^{5/2}\ \mathrm{kg\,m^{-3}}$$

☆ For a given T & comp., when material becomes degenerate

Eventually, e^- become rel. : $\Delta p \approx m_e c$

$$\rho \sim \frac{m_p}{(\Delta x)^3} \sim m_p\left(\frac{m_e c}{\hbar}\right)^3$$

Better calc. gives $\underline{\rho_{rel} \sim 10^9 \mu_e\ \mathrm{kg\,m^{-3}}}$

What about pressure? $P = (\gamma - 1)\,\varepsilon$

$E \approx pc \approx \hbar c/a$ so $\varepsilon \approx E/a^3 \approx \frac{\hbar c}{a^4}$

again, $\rho \sim \frac{m_p}{a^3}$ so $P \propto \rho^{4/3}$ & $\gamma = \frac{4}{3}$

$$\boxed{P_{rel} = \frac{(3\pi^2)^{1/3}\hbar c}{4}\left(\frac{\rho}{\mu_e m_n}\right)^{4/3}}$$

☆ Show Fig. 13.10

Same conditions for neutron matter if set $m_u = m_n$ & $\mu_e = 1$

E.o.S. indep. of T

Chandrasekhar Limit

Solve eqn of stellar structure,

see what happens when center becomes rel.

Solutions are polytropes w/ index n, where $\gamma = 1 + \frac{1}{n}$ $\left(P = K\rho^{\gamma} = K\rho^{(n+1)/n}\right)$

central density $\rho_c \propto R^{2n/(1-n)}$ $\rho \propto \rho_c R^3$

non-rel. → $\gamma = 5/3$, $n = \frac{3}{2}$, $\rho_c \propto R^{-6}$, $M \propto R^{-3}$

rel. → $\gamma = 4/3$, $n = 3$, $\rho_c \propto R^{-3}$, M indep. R

Stars shrink as they become more massive if deg.

Add mass to a non-rel. deg. star, radius ↓ & $\rho_c \uparrow \propto M^2$ until becomes rel., when it has to have a specific mass

$$M = \frac{5.836}{\mu_e^2} M_\odot = \boxed{1.46 M_\odot}$$

$\mu_e = 2$ b/c comp. is CO

$C \to 6e^- + 1C$: $\frac{12 m_u + 6 m_e}{6\ e^-(\#)} = 2$

Can use virial theorem to derive rel. limit

What is this?

$$U_{int} = \mathcal{E} V = P \frac{V}{(\gamma - 1)} \approx V \hbar c \left(\frac{\rho}{m_p}\right)^{4/3}$$

$$U_{int} = \frac{1}{2} |\Omega_{grav}| = \frac{1}{4} \frac{GM^2}{R} = V \hbar c \left(\frac{\rho}{m_p}\right)^{4/3}$$

$V \approx R^3$ & $\rho V = M$ → eliminate V + ρ

$$M \approx \frac{1}{m_p^2}\left(\frac{\hbar c}{G}\right)^{3/2} \approx 2 M_\odot$$

→ R cancels out: as $M\uparrow$, grav. pot. E always wins, $|\Omega_g| \propto M^2$ while $U \propto M^{4/3}$ & both scale the same w/ R → no equil. configuration

Can do for NS as well, find

$$M_{NS} \leq 5.73\, M_\odot$$

but doesn't include general rel. effects, which suggest $M_{NS} \lesssim 3 M_\odot$