# Review of Black Holes

## Non-Rotating Black Holes

**Schwarzschild Metric:**

$$(d\tau)^2 = \left(1 - \frac{2GM}{c^2 r}\right)(dt)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\frac{(dr)^2}{c^2} - \frac{r^2}{c^2}\left[(d\theta)^2 + \sin^2\theta(d\phi)^2\right]$$

Singularity at $r = 0$. *Event Horizon* at:

$$R_{\rm sch} = \frac{2GM}{c^2} \approx 3\left(\frac{M}{M_\odot}\right) \text{ km}$$

*Gravitational Time Dilation:*

$$dt = \left(1 - \frac{R_{\rm sch}}{r}\right)^{-1/2} d\tau \to \infty \quad \text{as} \quad r \to R_{\rm sch}$$

*Gravitational Redshift:*

$$1 + z_{\rm grav} \equiv \frac{\lambda_{\rm obs}}{\lambda_{\rm em}} = \left(1 - \frac{R_{\rm sch}}{r}\right)^{-1/2} \to \infty \quad \text{as} \quad r \to R_{\rm sch}$$

Not singular at $R_{\rm sch}$. Can find coordinate transformation which removes singularity. Example is *Kruskal Coordinates:*

$$\begin{aligned} u &= \left(\frac{r}{R_{\rm sch}} - 1\right)^{1/2} \exp\left(\frac{r}{2R_{\rm sch}}\right) \cosh\left(\frac{ct}{2R_{\rm sch}}\right) \\ v &= \left(\frac{r}{R_{\rm sch}} - 1\right)^{1/2} \exp\left(\frac{r}{2R_{\rm sch}}\right) \sinh\left(\frac{ct}{2R_{\rm sch}}\right) \end{aligned}$$

*Motions of particles with rest mass $m$:* Energy $E$ and angular momentum $L$ conserved. *Effective Potential:* Newtonian: $V_{\rm eff} = -GM/r + L^2/(2mr^2)$, with $E = (1/2)mv_r^2 + V_{\rm eff} \geq V_{\rm eff}$. For $L > 0$, motion always avoids $r = 0$.

*Schwarzschild Effective Potential:* $E = \sqrt{m^2c^2v_r^2 + V_{\rm eff}^2} \geq V_{\rm eff}$, with $v_r = dr/d\tau$ and

$$V_{\rm eff} = mc^2\sqrt{\left(1 - \frac{2GM}{c^2 r}\right)\left(1 + \frac{L^2}{m^2c^2r^2}\right)} = mc^2\sqrt{\left(1 - \frac{R_{\rm sch}}{r}\right)\left(1 + \frac{L^2}{m^2c^2r^2}\right)}$$

Last stable circular orbit at

$$R_{\rm lso} = \frac{6GM}{c^2} = 3R_{\rm sch}\,.$$

*Motions of photons:* Replace $L$ with *impact parameter* $b$. Conservation of energy requires that that $1/b^2 \geq V_{\rm phot}$, where

$$V_{\rm phot} = \frac{1}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) = \frac{1}{r^2}\left(1 - \frac{R_{\rm sch}}{r}\right)$$

Unstable circular orbit at $r = 3GM/c^2 = (3/2)R_{\rm sch}$.

## Rotating Black Holes

**Kerr Metric:**

$$\begin{aligned}(d\tau)^2 \;&=\; \left(1 - \frac{2GMr}{c^2\Sigma}\right)(dt)^2 + \frac{4GMar\sin^2\theta}{c^3\Sigma}\,(dt)\,(d\phi) - \frac{\Sigma}{\Delta}\,\frac{(dr)^2}{c^2} - \frac{\Sigma}{c^2}\,(d\theta)^2 \\ &\quad - \left(r^2 + a^2 + \frac{2GMa^2 r\sin^2\theta}{c^2\Sigma}\right)\frac{\sin^2\theta}{c^2}\,(d\phi)^2\,,\end{aligned}$$

where

$$a \equiv \frac{J}{Mc} \text{ (units length)} \qquad \Delta \equiv r^2 - 2GMr/c^2 + a^2 \qquad \Sigma \equiv r^2 + a^2\cos^2\theta$$

There is no solution if $a \geq GM/c^2$.

*Event Horizon* at larger root of $\Delta = 0$:

$$R_{\text{sch}} = \frac{GM}{c^2} + \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2}\,.$$

*Static Limit* or *Ergosphere* at larger root where $g_{tt} = 0$ at

$$R_{\text{static}} = \frac{GM}{c^2} + \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2\cos^2\theta}\,.$$