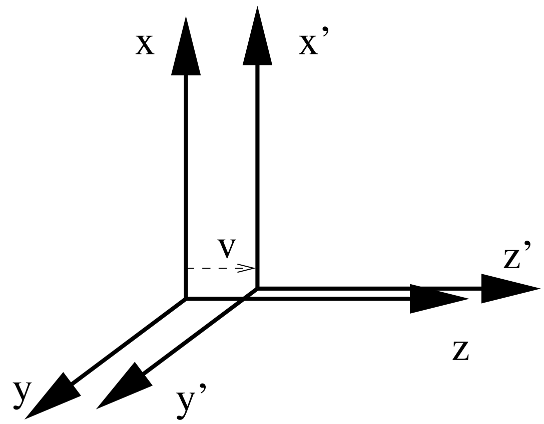
Review of Lorentz Transformations

Lorentz Transformations: Consider two inertial frames $(t, \vec{x})$ and $(t', \vec{x}')$. The primed frame moves at a speed v along the z-axis relative to the unprimed frame. Assume that initially $t = t' = 0$ and that the two coordinate frames coincide at $t = 0$. Then, the Lorentz transformation between the two frames is:



$$
\begin{aligned}
z' &= \gamma\,(z - vt) \\
x' &= x \\
y' &= y \\
t' &= \gamma\left(t - \frac{vz}{c^2}\right)
\end{aligned}
$$

where the Lorentz factor γ is defined as

$$\gamma \equiv \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2} .$$

Similar transformation apply to all other 4-vectors, including the energy-momentum 4-vector $p^\mu \equiv (E, \vec{p})$.

Energy and Momentum: The total energy of a particle moving at velocity v is

$$E = \gamma mc^2 \,,$$

where m is the rest mass. The kinetic energy is given by

$$K.E. = T = (\gamma - 1)mc^2 \approx \begin{cases} \frac{1}{2}mv^2 & v \ll c \\ \gamma mc^2 & \gamma \gg 1 \end{cases}$$

The magnitude of the momentum is given by

$$p = \gamma mv \approx \begin{cases} mv & v \ll c \\ E/c & \gamma \gg 1 \end{cases}$$

Doppler Shift: The wave 4-vector of light $k^\mu \equiv (\omega, \vec{k})$ also transforms similarly. Here, $k \equiv 2\pi/\lambda$ is the wavenumber, and $\omega = 2\pi\nu$ is the angular frequency, where λ is the wavelength and ν is the frequency in Hz. The transformation is:

$$
\begin{aligned}
k'_z &= \gamma\left(k_z - \frac{v\omega}{c^2}\right) \\
k'_x &= k_x \\
k'_y &= k_y \\
\omega' &= \gamma\left(\omega - v k_z\right) \, .
\end{aligned}
$$

One can also write the Doppler shift as

$$
\begin{aligned}
\omega' &= \gamma\omega\left(1 - \frac{v}{c}\cos\theta\right) \\
&= \omega/\left[\gamma\left(1 + \frac{v}{c}\cos\theta'\right)\right]
\end{aligned}
$$

where θ is the angle between $\vec{k}$ and $\vec{v}$ is the original frame $(t, \vec{x})$ and θ' is the angle between $\vec{k}'$ and $\vec{v}$ is the transformed frame $(t', \vec{x}')$.

Electromagnetic Fields: Let $\vec{E}_\parallel$ and $\vec{B}_\parallel$ be the components of the electric and magnetic fields parallel to $\vec{v}$, and let $\vec{E}_\perp$ and $\vec{B}_\perp$ be the perpendicular components. The fields transform as

$$
\begin{aligned}
\vec{E}'_\parallel &= \vec{E}_\parallel \\
\vec{B}'_\parallel &= \vec{B}_\parallel \\
\vec{E}'_\perp &= \gamma\left(\vec{E}_\perp + \frac{\vec{v}\times\vec{B}}{c}\right) \\
\vec{B}'_\perp &= \gamma\left(\vec{B}_\perp - \frac{\vec{v}\times\vec{E}}{c}\right)
\end{aligned}
$$