Homework 4

Due date: March 18, 2019 (Mon, in class). No late homework.

1. (60 pts) Aberration of Light from Fields Transformation

In an inertial frame of reference K, a monochromatic plane electromagnetic wave is propagating along $\mathbf{k}$ direction. The $\mathbf{k}$ direction is in the x–y plane and at an angle θ to the $+x$–axis. So the wave vector has components $(k_x, k_y, k_z) = (k\cos\theta, k\sin\theta, 0)$. The wave is linearly polarized with the electric field in the x–y plane.

(a) Write down an expression for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ of the wave. You can assume an arbitrary initial phase of the wave at $\mathbf{r} = 0$ and $t = 0$. Or you can completely ignore the phase (which is Lorentz invariant and does not play a role in the following questions).

(b) Transform the above $\mathbf{E}$ and $\mathbf{B}$ fields to an inertial frame of reference K' moving at velocity v along the $+x$-axis. Based on the results, show that in K' the wave is propagating at an angle θ' to the $+x'$–axis, with

$$\sin\theta' = \frac{\sin\theta}{\gamma\,(1-\beta\cos\theta)}, \tag{1}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$ have their usual meanings. That is, we obtain the aberration of light by considering fields transformation.

(c) If the flux of the wave (energy per unit time passing through a unit area perpendicular to the propagation direction) in the K frame is F, show that the flux F' in the K' frame is

$$F' = F\gamma^2\,(1-\beta\cos\theta)^2\,. \tag{2}$$

(**hint:** think about the Poynting flux).

2. (40 pts) Condition for Pair Production

In the lab frame K, two photons of energy E_1 and E_2 traveling in opposite directions have a head-on collision. There exists an inertial frame of reference $K_{\rm cm}$, called the center-of-momentum frame, where the momenta of the two photons have the same magnitude but in opposite directions.

(a) What is the velocity (or the $\beta \equiv v/c$ factor) of frame $K_{\rm cm}$ relative to frame K? (Without losing generality, you can assume $E_1 \geq E_2$.)

(b) In the center-of-momentum frame $K_{\rm cm}$, if the total energy of the two photons exceeds twice the rest energy of an electron (i.e., $2mc^2$), the collision of the two photons can lead to electron-positron pair production. Express the pair production condition in terms of the energy E_1 and E_2 in the lab frame K. That is, your results should only consist of E_1, E_2, m, c, and other physical and numerical constants.

3. (100 pts) Compton Drag on an Electron due to Scattering of Radiation

An electron is moving with velocity $v = \beta c$ (not necessarily non-relativistic), relative to a frame where there is a uniform blackbody radiation at temperature T (e.g., the Cosmic Microwave Background). The temperature is sufficiently low, so Thomson scattering is applicable in the rest-frame of the electron. Because of Thomson scattering, the electron will slow down.

Show that the rate of the change of the momentum of the electron in the frame of the uniform blackbody radiation is

$$\frac{dp}{dt} = -\frac{4}{3}\gamma^2 \beta \sigma_T a T^4, \tag{3}$$

where σ_T is the Thomson scattering cross section and a is the radiation density constant.

[**N.B.** This problem intends to be solved *without* the results of the Inverse Compton scattering. So the results based on Inverse Compton scattering (e.g., Equation 7.16a in Rybicki&Lightman in Chapter 7) should **NOT** be used. What you need to know to solve this problem are only Thompson scattering and Lorentz tranformation, and this problem would help you review the materials and establish the picture.]

[**Hint:** There can be different ways of solving this problem. You can follow your own idea. Here is just one possible way of doing it: Think about the way the radiation looks like in the (instantaneous) rest-frame of the electron. Also note that Thomson scattering has forward-backward symmetry, i.e., for the incident radiation at any given direction, the scattered radiation has zero average momentum in the rest-frame of the electron. You may also want to look at Problem 4.13 in Rybicki&Lightman and use some of the results.

If you are still not sure about how to solve it after thinking it over, see the next page for more detailed hints and the suggested steps to follow.]

[**More hints.** Let's first consider what happens in the rest-frame of the electron, and we denote all the quantities with $'$'s in this frame.

(1) Write down the temperature $T'(\theta')$ of the incident photons propagating in the direction that has an angle θ' with respect to the electron's velocity direction (you will find it helpful to go through Problem 4.13 in Rybicki&Lightman).

(2) Suppose that the radiation coming from one direction is scattered by the electron. Since Thomson scattering has the forward-backward symmetry, the scattered radiation has zero total momentum, which means that the momentum of the radiation intercepted by the electron (or scattered by the electron) is transferred to the electron. Given the symmetry around the electron's velocity direction, we only need to consider the component of the incident radiation momentum parallel to the motion of the electron. Write down the energy flux $d\mathcal{F}'_{\nu'}$ of the incident radiation (energy per unit time per unit area per frequency interval) in a solid angle $d\Omega'$ in the direction θ'. What is the energy flux $d\mathcal{F}'$ (energy per unit time per unit area) after integrated over frequency? What is the corresponding momentum flux dF'?

(3) What is the component of the total (integrated over all solid angle) incident radiation momentum flux $F'_{\|}$ parallel to the direction of the electron's motion? If you work out everything correctly, the result will be a function of electron velocity v and blackbody temperature T.

(4) A fraction of the above incident radiation momentum flux will be intercepted by the electron and scattered. The net momentum change of the radiation [see (2)] will correspond to that of the electron. Write down the momentum change of the electron per unit time $dp'_{\|}/dt'$.

(5) The result in (4) is in the rest-frame of the electron, and we need to transfer it to the frame where the radiation is isotropic. Find out $dp_{\|}$ and dt (by making use of the transformation of four-momentum and time), respectively, and then derive dp/dt as the result.]